Revenue sharing at music streaming platforms*

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Abstract

We study the problem of sharing the revenues raised from subscriptions to music streaming platforms among content providers. We provide direct, axiomatic and game-theoretical foundations for two focal (and somewhat polar) methods widely used in practice: *pro-rata* and *user-centric*. The former rewards artists proportionally to their number of total streams. With the latter, each user's subscription fee is proportionally divided among the artists streamed by that user. We also provide foundations for a family of methods compromising among the previous two, which addresses the rising concern in the music industry to explore new streaming models that better align the interests of artists, fans and streaming services.

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1 Introduction

Steve Jobs unveiled the iPod on October 23, 2001. As he put it himself a few years later, that event *didn't just change the way we all listen to music, it changed the entire music industry.* And it is difficult to deny such a statement. Because digital music had started to emerge at the end of the 20th century, mostly thanks to the advent of peer-to-peer (P2P) technologies. But successful file-sharing platforms, such as Napster, were under scrutiny by the music industry, which aggressively pursued stronger copyright enforcement and regulations (e.g., Bhattacharjee et al., 2007). In that environment, Apple managed to persuade record companies to sale individual tracks for 99 cents. Gradually, the industry found a new way to stay profitable and even embraced new technology advances like streaming. A decade later, Spotify, an early forerunner music streaming platform, announced a customer base of 1 million paying subscribers across Europe, and was officially launched in the US. Alternative services (such as Beats Music, Amazon Music Unlimited and Google Play Music All-Access, as well as a new Apple Music service) also emerged in the market. According to Statista, revenue in the Music Streaming market is projected to reach US$25.84 billions in 2023. The number of users is expected to amount to 1.1 billion by 2027.[1]

Streaming platforms generate massive amounts of revenue nowadays. Aleei et al. (2022) argue that "streaming services generated $4.3 billion in the first half of 2019 (with 77% of that coming from paid subscriptions)". Users typically pay a fixed (monthly) amount to freely access their libraries. A common practice for platforms is to distribute around 70% of the revenue received from subscriptions among artists (e.g., Meyn et al., 2023). Platforms also raise money from other sources (for instance, advertisements) but the most important source are subscriptions and we shall concentrate on them here. An ensuing interesting problem is to allocate the corresponding part of those revenues among participating artists, based on their streaming times. This will be the object of study in this paper.

We introduce a stylized model, whose ingredients are three: a group of artists, a group of users and a matrix indicating the streaming times each user played each artist. Based on this input, a popularity **index**, which measures the importance of each artist, is constructed. The reward received by each artist from the revenues generated in each problem is based on such a popularity index. The most frequent indices are the so called **pro-rata** index, which renders artists rewarded proportionally to the total number of streams and the so called **user-centric** index, which renders artists rewarded so that the revenue generated by each user is divided among the artists listened by the user proportionally to the total number of streams.[2] Then, the amount received by each artist is computed by adding the amounts obtained from each user.

[1] https://www.statista.com/outlook/dmo/digital-media/digital-music/music-streaming/worldwide

[2] Typically, a user pays around $10 per month. As 70% of the amount generated by users is devoted to pay artists, we could just consider in our model each user pays $7. For ease of exposition, we shall assume that each user pays a normalized amount of 1 to have unlimited access to the contents in the streaming platform, during the period of time (usually, a month). Note also that, in practice, it is considered that an artist obtained a streaming unit when a user played a song from that artist for at least 30 seconds (e.g., Meyn et al., 2023). Thus, the entry of the (streaming) matrix t_{ij} would be the number of streaming units artist i obtained from user j during one month. There is, nevertheless, a debate in the industry about how streams should be computed. Another alternative is a remuneration based on per-second usage (e.g., Meyn et al., 2023). Our theoretical model is general and allows us to accommodate other ways of measuring streams.

We take several approaches to analyze our model. In the first (axiomatic) approach, we present axioms that formalize normatively appealing principles for popularity indices. Some convey structural ideas that reflect operational features of the index. For instance, *additivity* says that if we can present a problem as the sum of smaller problems (such as in, say, multinational platforms), then the index in the original problem should coincide with the sum of the indices in the smaller problems. And *homogeneity* says that the index should reflect accordingly the cases in which each user has reproduced content from a given artist a certain times more than content from another artist. Other axioms (*equal individual impact of similar users* and *equal global impact of users*) model alternative forms of *marginalism*, i.e., the impact of extra users in the platform. And we also consider axioms reflecting concerns for fairmess: *reasonable lower bounds* and *click-fraud-proofness*. The first one states that artists should at least receive the amount paid by the users that only played content provided by them. The second one states that if a user changes streaming times, then the amount received by each artist could not change more than the subscription paid by that user. We explore how the main indices perform with respect to all these axioms, but our main result in this axiomatic approach to the problem (Theorem 1) states three characterization results. First, the combination of *additivity*, *homogeneity* and *equal individual impact of similar users* characterizes the *pro-rata* index. Second, the combination of *additivity*, *homogeneity* and *equal global impact of users* characterizes the *user-centric* index. Third, the combination of *additivity* and *homogeneity* characterizes a family of *weighted* indices that compromise between the *pro-rata* index and the *user-centric* index, upon assigning each artist the weighted aggregation of streamings, with the weight depending on the user and her streaming profile.

We then take two other indirect approaches to deal with streaming problems. In both of them, streaming problems are associated to other problems that have already been studied in the literature, and we then import well-known solutions for those problems to solve streaming problems.

First, we associate each streaming problem with a *cooperative game with transferable utility*, where the worth of each set of artists is defined as the amount paid by the users that have only played content from those artists. Such a game is convex, i.e., the incentives to join a coalition increase as the coalition grows. It is well known that the core of a convex game is quite large and we fully characterize it in our Theorem 2. In words, a core allocation imposes that the amount paid by each user is divided in any way among the artists listened by this user. Each artist then receives the sum (over all users) of the corresponding amounts. We show that the *pro-rata* index does not always guarantee allocations within the core, whereas the *user-centric* index always does so.

Second, we associate each streaming problem with a *claims problem*, where the users are identified as the *issues* in the claims problem. Then, problems can be solved in two stages. Either focussing on each issue (user) first and agents (artists) afterwards, or viceversa. We show (Theorem 3) how the indices we consider for streaming problems can be rationalized as two-stage claims rules. More precisely, the *pro-rata* index and the *user-centric* index can be rationalized as *weighted proportional rules* (although there is no bijection between this family of rules and the weighted indices for streaming problems). Furthermore, the allcation rules these two indices induce can also be described as two-stage (claims) rules where we first decide the importance of each user and then the importance of each artist for each user, which is computed as the sum over all users.

1.1 Related literature

The closest paper to ours is Alaei et al. (2022). They also consider the same streaming platforms that we consider here, which generate revenues by charging users a subscription fee for unlimited access to the content and compensate artists through an allocation rule. They also assume that users are heterogeneous in both their overall consumption and the distribution of their consumption over different artists, but they model this by referring to the probability (per usage) that each user type wants to consume the content of each artist. In our case, we talk about the number of times a user plays an artist. But, leaving this minor aspect aside, our models are essentially equivalent. They are also concerned with the pro-rata and user-centric revenue allocation methods, but focus on characterizing when these two methods can sustain a set of artists on the platform, as well as comparing them from both the platform's and the artists' perspectives. In particular, they show that, despite the cross-subsidization between low- and high-streaming-volume users, the pro-rata method can be preferred by both the platform and the artists.[3] More precisely, they show that artists who are predominantly listened to by users whose overall streaming volume (consumption) is high receive higher payments with the pro-rata allocation than with the user-centric allocation. Consequently, if there is an artist who is extremely popular with users who have high consumption, the pro-rata rule is preferred from the platform's viewpoint (actually, the platform might not even be profitable with the user-centric rule in this scenario). More generally, the analysis in Alaei et al. (2022) suggests that *"pro-rata should be the artist's compensation rule of choice if the users' behavior points to correlation between artists' popularity and consumption among users and/or if the platform might not be able to estimate heterogeneity of artists' popularity and consumption among its users"*.[4]

Our analysis complements Alaei et al. (2022) in several ways. We provide normative foundations for both methods, as well as game-theoretical and other indirect approaches to the problem. And we can safely conclude from our analysis that there exist powerful arguments to prefer the user-centric method to the pro-rata method. We acknowledge that the pro-rata method made sense when the technology to divide revenue by individual users' streams did not exist. But that is no longer the case and, consequently, a user-centric method that distributes revenue based on each listener's individual streams seems to be more reasonable (based on the arguments we provide in our analysis). But we are not the first ones suggesting that. Haampland et al. (2022) resort to the data provided by a leading music streaming platform in France (with 427 million streams coming from more than 140,000 unique users during six months). They obtain from their empirical analysis that, when compared with the pro-rata method, the user-centric method renders consumers' choices better aligned with revenue sharing, but also that it favors organic streams with respect to curated streams, it reduces the superstar phenomenon, and it moderates the potential bias in curated streams. Dimont (2017) also endorses switching from a pro-rata method to a user-centric method from a legal perspective. Muikku (2017) finds that as the overall stream count decreases, the revenue difference between the user-centric and the pro-rata methods increases. The latter favours artists and tracks, which get the biggest amount of played streams regardless if

[3] Lei (2023) also obtains that the pro-rata method can outperform the user-centric method in a simple model with only two artists.

[4] Alaei et al. (2022) also obtain interesting computational results for this problem. For instance, they show that the platform's problem of selecting an optimal portfolio of artists is NP-complete, but develop a polynomial time approximation scheme for the optimal platform's profit. We do not consider computational aspects in this paper.

they are created by a large number of users with few plays or a smaller number of users who have played them repeatedly. The former favours artists with smaller number of streams, especially when the overall stream count is smaller. However, the results depend on the cumulative effects of both individual and user groups' listening habits. Meyn et al. (2023) investigate the monetary consequences of the switch from the pro-rata method to the user-centric method. Using individual-level data from 3,326 participants and data from Spotify Web API, they find a substantial reallocation (mostly shifting revenue from mainstream to niche genres) of nearly €170 million per year at Spotify, driven by the song length as well as the payments per listening time.

Beyond the specific issue of sharing the revenue raised from subscriptions to (online music) platforms, the literature (at the intersection of several fields such as economics, management, marketing and law) has addressed other issues regarding the effect that streaming has on the music industry. For instance, Aguiar and Waldfogel (2018) find that although streaming displaces sales, it also displaces music piracy. Aguiar and Waldfogel (2021) highlight the market power of some platforms (especially, Spotify) in the online music market (via their via platform-operated playlists, which act as promotion channels). Datta et al., (2018) analyze the effects of the gradual adoption of streaming services in the music industry. They find that adoption of streaming leads to very large increases in the quantity and diversity of consumption in the first months after adoption. And, although the effects attenuate over time, adopters continue playing substantially more, and more diverse, music. Jain and Qian (2021) focus on another major source of revenue for platforms; namely, advertising, which depends on the number of active customers using the platforms. Their emphasis is on how such sharing incentives are affected by the nature of competition among various producers, the size of the customer base, and the type of customers. They show that increased producer competition can lead to higher compensation for the producers, higher content quality, and higher producers' profits. And, more generally, Waldfogel (2017) surveyed the literature dealing with digitalization and its positive effects in different art industries (not only music, but also films, literature and television), whereas Rietveld and Schilling (2020) reviewed the literature (comprising the last three decades) on platform competition.

Our work also relates to the industrial organization literature dealing with bundling, which can be traced back to Adams and Yellen (1976). Bundling products is typically an effective mechamism to increase revenue with respect to selling products independently. Examples abounded in real life even before the advent of digital content platforms (e.g., Ginsburgh and Zang, 2003; Bergantiños and Moreno-Ternero, 2015). Online bundling might nevertheless have a different nature, as it often occurs in a distribution channel where a downstream firm (retailer) adopts a bundling strategy involving products made by separate and multiple upstream firms (manufacturers). This sort of practice not only applies to the platforms offering entertainment products, such as the ones we analyze here, but also to other information goods such as online services (e.g., Bhargava, 2013).[5] The complex relationship between the independent price of each product and the bundled price renders the problem of sharing the revenue from periodic charges to unlimited streaming among the participating agents a complex one too. Nevertheless, we believe our results could shed light on that problem.

[5] Geng et al. (2005) point out how consumers' average value for consuming a stream of information goods (such as streaming music, or other online entertainment) declines with the number consumed and provide basic guidelines for optimal bundling marketing strategies in such a case.

Our paper (especially, Section 3) is also connected to the sizable axiomatic literature on resource allocation or, more generally, the axiomatics of economic design (e.g., Thomson, 2023). A variety of axioms formalizing principles with ethical or operational appeal for resource allocation have been introduced in economic theory for more than half a century. Instances are axioms formalizing no-envy, impartiality, priority, or solidarity (e.g., Foley, 1967; Moreno-Ternero and Roemer, 2006; Thomson, 2023). They have contributed to our understanding of normative issues concerning the allocation of goods and services. Our paper extends the scope of this literature in order to deal with a special and somewhat new type of goods and services that have arisen in the current digital era. But the characterizations we provide in Section 3 are also reminiscent of other recent characterizations in the axiomatic literature on resource allocation. For instance, Bergantiños and Moreno-Ternero (2020) analyze the problem of sharing the (collectively raised) revenues from sportscast.[6] The input in their case is the audience matrix indicating the number of viewers of each game between two clubs participating in a given competition (thus, similar to the input of the streaming problems we analyze in this paper). And they also characterize two focal rules to share those (collectively raised) revenues.[7] Singal et al. (2022) also develop an axiomatic framework to study the related problem of attribution in online advertising; namely, assessing the contribution of individual advertiser actions to eventual conversion.[8] Somewhat related, Flores-Szwagrzak and Treibich (2020) introduce an innovative productivity index that disentangles individual from collaborative productivity. This index defines an individual's productivity score as the sum of her credit over all the projects she has contributed to. Credit on each project is allocated proportionally to the score of each teammate and the scores are thus determined endogenously and simultaneously (solving a fixed point problem). Martinez and Moreno-Ternero (2022) characterize a family of pandemic performance indicators arising from a weighted average of the incidence rate, morbidity rate and mortality rate. Previously, Hougaard and Moulin (2014) studied how to share the cost of finitely many public goods (items) among users with different needs. They characterized a family of cost ratios based on simple liability indices, one for each agent and for each item, measuring the relative worth of this item across agents, and generating cost allocation rules additive in costs. And, more recently, Gudmundsson et al. (2023) also take the axiomatic approach to analyze the problem of sharing sequentially triggered losses. They characterize a class of fixed-fraction rules, which strike a balance between incentives for accident prevention and fairness to assign liabilities. Finally, Gonçalves-Dosantos et al. (2023) also take the axiomatic approach (but not the game-theoretical, or another indirect approach) to study general content streaming platforms (such as, for instance, Netflix). As music streaming platforms have some specific characteristics that are not necessarily extended to general content streaming platforms, most of the axioms considered in both papers are of a different nature. Characterization results are thus genuinely different.[9]

To conclude with the introduction, we mention that our paper is also related to the literature on cooperative

[6] Dietzenbacher and Kondratev (2023) study the related problem of allocating a prize endowment among competitors, when their ranking is known.

[7] Families of rules compromising among the two rules have also been considered in the literature (e.g., Bergantiños and Moreno-Ternero, 2021, 2022).

[8] Both Bergantiños and Moreno-Ternero (2020) and Singal et al. (2022) also explore the game-theoretical approach to solve their problems, as we do in this paper.

[9] As a matter of fact, in all of our characterizations we use axioms that do not appear in Gonçalves-Dosantos et al. (2023), whereas in all of the characterizations they provide, they use some axioms that do not appear in our paper.

game theory. There is a tradition of analyzing problems involving agents' cooperation with a game-theoretical approach. Classical instances are the so-called airport problems (e.g., Littlechild and Owen, 1973), in which the cost of a runway has to be shared among different types of airplanes, bankruptcy problems from the Talmud (e.g., Aumann and Maschler, 1985), telecommunications problems and the rerouting of international telephone calls (e.g., van den Nouweland et al., 1996), or modern transportation systems (e.g., Krajewska et al., 2008). One of the approaches we take in this paper (Section 4) is precisely this one. The (cooperative) game we associate to streaming problems is convex and, thus, it has a large core encompassing allocations guaranteeing participation constraints. We fully characterize such a core and show that the pro-rata method can generate allocations that violate these constraints. We also show that the resulting axiom of core-selection is key to characterize the user-centric method. The combination of these two results is another powerful argument we provide to support the user-centric method with respect to the pro-rata method.

The game-theoretical approach is an indirect way of solving streaming problems. Another indirect approach suggests to solve them via associating a claims problem (rather than a cooperative game). The problem of adjudicating conflicting claims (in short, claims problem) models a basic situation in which an endowment is allocated among agents who have claims on it, and the available amount is not enough to fully honor all claims. This is a classic problem that can be traced back to ancient sources, such as Aristotle and the Talmud, although is formal treatment is somewhat recent (e.g., O'Neill, 1982; Thomson, 2019). Ju et al. (2007) generalize these problems to account for multiple issues.[10] We show that some of the two-stage rules from generalized claims problems can rationalize some of the allocation rules we consider to solve streaming problems.

The rest of the paper is organized as follows. In Section 2, we introduce our model and main concepts for our analysis of streaming problems. In Section 3, we present the axiomatic approach to our problems including the main characterization results we obtain. In Section 4, we explore the game-theoretical approach to our problem. In Section 5, we explore another indirect approach to solve our problems based on claims problems. Section 6 concludes. Most of the proofs have been postponed to an appendix.

2 Streaming problems

Let $N = \{1, ..., n\}$ denote a set of artists and $M = \{1, ..., m\}$ a set of users. For each pair $i \in N, j \in M$, let t_{ij} denote the times user j played (via streaming) contents uploaded by artist i in the platform, briefly streams, during a certain period of time (e.g., month). Let $t = (t_{ij})_{i \in N, j \in M}$ denote the corresponding matrix encompassing all playing times. A **streaming problem** is $P = (N, M, t)$.[11] The set of problems so defined is denoted by $\mathcal{P}$. For each $j \in M$, we denote by t^{-j} the matrix obtained from t by removing the column corresponding to user j.

[10] Csóka and Herings (2018, 2021) also studied recently a different generalization of claims problems to deal with financial networks, as pioneered by Eisenberg and Noe (2001). See also Calleja and Llerena (2023).

[11] As mentioned at the Introduction, and for ease of notation, we normalize the amount paid by each user to 1. Thus, the amount to be divided among artists in a problem (N, M, t) is just m, the number of users.

For each artist $i \in N$,

$$T_i(N, M, t) = \sum_{j \in M} t_{ij},$$

denotes the total times i was played. Likewise, for each user $j \in M$

$$T^j(N, M, t) = \sum_{i \in N} t_{ij},$$

denotes the total times j played content.

We define the set of fans of each artist as the set of users who have played content from the artist at least once. Formally, $F : N \to M$ is such that for each $i \in N$,

$$F_i(N, M, t) = \{j \in M : t_{ij} > 0\}.$$

Similarly, we define the list of artists of a user as those from which the user has played content at least once. Formally, $L : M \to N$ such that for each $j \in M$,

$$L^j(N, M, t) = \{i \in N : t_{ij} > 0\}.$$

The profile of user j is defined as the streaming vector associated to such a user. Namely,

$$t_{.j}(N, M, t) = (t_{ij})_{i \in N}.$$

When no confusion arises we write T_i instead of $T_i(N, M, t)$, T^j instead of $T^j(N, M, t)$, F_i instead of $F_i(N, M, t)$, L^j instead of $L^j(N, M, t)$, and $t_{.j}$ instead of $t_{.j}(N, M, t)$.

We illustrate our model with a basic example that will surface throughout our ensuing analysis. Assume two users (a, b) join a platform to listen to their favorite artists (1,2). Assume artist 1 is listened by user a, whereas artist 2 is listened by user b. This situation can be included in our theoretical model as follows.

Example 1 *Let* $N = \{1, 2\}$, $M = \{a, b\}$, *and*

$$t = \begin{pmatrix} 10 & 0 \\ 0 & 90 \end{pmatrix}.$$

A popularity **index** (I) for streaming problems is a mapping that measures the importance of each artist in each problem. Formally, $I : \mathcal{P} \to \mathbb{R}^N_+$ and, for each pair $i, j \in N$, $I_i(N, M, t) \geq I_j(N, M, t)$ if and only if i is at least as important as j at problem (N, M, t). We assume that $\sum_{i \in N} I_i(N, M, t) > 0$.

The reward received by each artist $i \in N$ from the revenues generated in each problem (m because the amount paid by each user has been normalized to 1) is based on the importance of that artist in that problem. Formally,

$$R^I_i(N, M, t) = \frac{I_i(N, M, t)}{\sum_{i' \in N} I_{i'}(N, M, t)} m.$$

Note that any positive linear transformation of a given index generates the same allocation of rewards. Formally, for each $\lambda > 0$ and each index I, $R^{\lambda I} \equiv R^I$. Thus, in what follows, we shall slightly abuse language to identify an index with all its positive linear transformations too.

The index used in most of the platforms is the so called **pro-rata** index, which simply measures importance by the total number of streams. Formally, for each problem $(N, M, t) \in \mathcal{P}$ and each artist $i \in N$,

$$P_i(N, M, t) = T_i = \sum_{j \in M} t_{ij}.$$

Thus, the amount received by each artist $i \in N$ under P is

$$R_i^P(N, M, t) = \frac{T_i}{\sum_{j \in N} T_j} m.$$

Another basic index is the so called **user-centric** index. All users have the same importance (normalized to 1). The importance of each user is divided among the artists listened by this user proportionally to the total number of streams. Then, the importance of each artist is the sum, over all users, of the importance of the artist given by each user. Formally, for each problem (N, M, t) and each artist $i \in N$,

$$U_i(N, M, t) = \sum_{j \in M} \frac{t_{ij}}{T^j}.$$

As

$$\sum_{i \in N} \sum_{j \in M} \frac{t_{ij}}{T^j} = \sum_{j \in M} \sum_{i \in N} \frac{t_{ij}}{T^j} = \sum_{j \in M} 1 = m,$$

it follows that the amount received by each artist $i \in N$ from this index is precisely

$$R_i^U(N, M, t) = U_i(N, M, t).$$

In Example 1, we then have the following:

	$R_i^P(N, M, t)$	$R_i^U(N, M, t)$	T_i
1	0.2	1	10
2	1.8	1	90

That is, pro-rata gives to artist 2 much more than to artist 1, whereas the user-centric gives the same to both artists (which sounds more reasonable in this example). This might illustrate why some platforms are moving from pro-rata to user-centric.[12]

There is also room for alternative payment schemes beyond the previous two. For instance, Meyn et al.., (2022) write "remuneration based on quality ratings, or a combination of user-centric and pro-rata remuneration". And, also, "in addition to pro-rata and user-centric, the distribution parameters (e.g., the unit of distribution) must be investigated further". Partly accounting for this motivation, we now introduce a family of indices that allow us to consider alternative payment schemes compromising between the previous two.

A weight system is a function $\omega : M \times \mathbb{N}_+^N \to \mathbb{R}$ such that for each $j \in M$ and each $x \in \mathbb{N}_+^N$, $\omega(j, x) > 0$. For each weight system ω, its **weighted index** I^ω is defined so that for each (N, M, t) and each $i \in N$,

$$I_i^\omega(N, M, t) = \sum_{j \in M} w(j, t_{\cdot j}) t_{ij}.$$

[12] For instance, SoundCloud did so in 2021 (e.g., Ingham, 2021) whereas Deezer started an online campaign promoting user-centric (e.g., Deezer, 2019). Muikku (2017), Alaei et al. (2022), Haampland et al. (2022), and Meyn et al. (2023), among others, have compared the two indices on different grounds.

The index of each artist is obtained as the sum, over all users, of the streams of the user weighted by a factor that depends on the user (allowing, for instance, for popular users to have more importance than unknown users) and the streaming profile of the user (allowing, for instance, for more active users to have more importance than less active users). The pro-rata index is the weighted index associated with $\omega(j,x)=1$ for each $j\in M$, and each $x\in\mathbb{N}_{+}^{N}$ (namely, the users and the profile do not matter) and the user-centric index is the weighted index associated with $\omega(j,x)=\frac{1}{\sum_{i\in N}x_i}$ for each $j\in M$, and each $x\in\mathbb{R}_{+}^{N}$ (namely, the user does not matter but the profile matters).

Notice that under pro-rata the importance of each user to all artists, namely $\sum_{i\in N}w(j,t_{\cdot j})\,t_{ij}=T^j$, is proportional to his/her total streams. Under user-centric all users have the same importance, namely $\sum_{i\in N}w(j,t_{\cdot j})\,t_{ij}=1$. We can consider weight systems reflecting that all users have a minimum importance and also that users with more streamings contribute more, but with an upper bound. For instance, for each $j\in M$,

$$w(j,x)=\begin{cases}\frac{1}{\sum_{i=1}^{n}x_i} & \text{if } \sum_{i=1}^{n}x_i\le\alpha,\\ \frac{1}{\alpha} & \text{if } \alpha<\sum_{i=1}^{n}x_i\le\beta,\\ \frac{\beta}{\alpha\sum_{i=1}^{n}x_i} & \text{if } \sum_{i=1}^{n}x_i>\beta.\end{cases}$$

Notice that $\omega(j,x)$ does not depend on j. Thus, we only distinguish users through the streaming times. More precisely, all users with a number of streams below α have the same importance as in the user-centric index. Namely, for each $j\in M$,

$$\sum_{i\in N}w(j,t_{\cdot j})\,t_{ij}=1.$$

Users with a number of streams between α and β have a proportional importance to the streams.Namely, for each $j\in M$,

$$\sum_{i\in N}w(j,t_{\cdot j})\,t_{ij}=\frac{T^j}{\alpha}.$$

Finally, users with a number of streams above β have the same importance, given by the thresholds α and β. Namely, for each $j\in M$,

$$\sum_{i\in N}w(j,t_{\cdot j})\,t_{ij}=\frac{\beta}{\alpha}.$$

We now revisit Example 1, adding a new user (c) with 5 streams for artist 1 and 35 streams for artist 2. Besides, let $\alpha=20$ and $\beta=60$. Then, the weigthed index $I^{\alpha,\beta}$ defined as above is

$$\begin{aligned}I_1^{\alpha,\beta}(N,M,t) &= \left(\frac{1}{10}\right)10+\left(\frac{60}{20*90}\right)0+\left(\frac{1}{20}\right)5=1.25 \text{ and}\\ I_2^{\alpha,\beta}(N,M,t) &= \left(\frac{1}{10}\right)0+\left(\frac{60}{20*90}\right)90+\left(\frac{1}{20}\right)35=4.75.\end{aligned}$$

In this case, $P(N,M,t)=(15,125)$ whereas $U(N,M,t)=(1.125,1.875)$. Then, the rewards induced for artists are

	$R_i^P(N,M,t)$	$R_i^U(N,M,t)$	$R_i^{I^{\alpha,\beta}}(N,M,t)$
1	0.3	1.1	0.6
2	2.7	1.9	2.4

We observe that $I^{\alpha,\beta}$ yields an allocation in between those the pro-rata and user-centric yield.

3 An axiomatic approach

In this section, we take the axiomatic approach to solve streaming problems. That is, we formalize axioms of indices that model principles with normative (ethical or operational) appeal. Some of them will echo the concern that artists are paid fairly.[13] Some others are inspired from related discussions in the music industry or from the literature on resource allocation.

To introduce the first axiom, which is standard in axiomatic work, assume that each user has played artist i a certain times more than artist i'. Then, the index should preserve that ratio. Formally,

Homogeneity. For each $(N, M, t) \in \mathcal{P}$, each pair $i, i' \in N$, and each $\lambda \geq 0$ such that $t_{ij} = \lambda t_{i'j}$ for all $j \in M$,

$$I_i(N, M, t) = \lambda I_{i'}(N, M, t).$$

The second axiom is also a standard axiom in resource allocation and says that if we can divide a problem in the sum of two smaller problems, then the solution to the original problem should be the sum of the solutions in the two smaller problems. The intuition in streaming problems is the following. Suppose that a platform operates in several countries. Then, we can reward artists in two ways. First, we consider all countries in the same market and we allocate artists according with the streams in all countries. Second, we consider each country as a different market. Thus, each artist receives an allocation in each country according with the streams in this country. The total allocation to an artist is the sum over all countries. Additivity says that both ways should coincide. Formally,

Additivity. For each trio $\left(N, M^1, t^1\right), \left(N, M^2, t^2\right), (N, M, t) \in \mathcal{P}$ such that $M = M^1 \cup M^2$, $t_{ij} = t^1_{ij}$ when $j \in M^1$ and $t_{ij} = t^2_{ij}$ when $j \in M^2$,

$$I(N, M, t) = I\left(N, M^1, t^1\right) + I\left(N, M^2, t^2\right).$$

We now consider two alternative forms of modeling the impact of extra users.

Assume that two users (say, j and j') have the same streams on artist i. We can then consider that both users are similar for artist i. Then, both users should have the same impact over this artist.[14] Formally,

Equal individual impact of similar users. For each $(N, M, t) \in \mathcal{P}$, each $i \in N$, and each pair $j, j' \in M$ such that $t_{ij} = t_{ij'}$,

$$I_i\left(N, M \backslash \{j\}, t^{-j}\right) = I_i\left(N, M \backslash \{j'\}, t^{-j'}\right).$$

It is also often argued that as all users pay the same, all users should have the same impact on the index. We formalize this idea as follows:

Equal global impact of users. For each $(N, M, t) \in \mathcal{P}$ and each pair $j, j' \in M$,

$$\sum_{i \in N} I_i\left(N, M \backslash \{j\}, t^{-j}\right) = \sum_{i \in N} I_i\left(N, M \backslash \{j'\}, t^{-j'}\right).$$

The next result states the characterizations we obtain combining the previous axioms.

[13] Haamland et al., (2022) provide an interesting discussion about this issue.

[14] A rationale for this axiom is provided by Page and Safir (2018a), who write "some listeners may find it inequitable that a given track be awarded significantly different per-stream values".

Theorem 1 *The following statements hold:*

(a) *An index satisfies homogeneity, and additivity if and only if it is a weighted index.*

(b) *An index satisfies homogeneity, additivity, and equal individual impact of similar users if and only if it is the pro-rata index.*

(c) *An index satisfies homogeneity, additivity, and equal global impact of users if and only if it is the user-centric index.*

Theorem 1, whose proof is postponed to the appendix, provides normative grounds for the indices considered above. Statement (a) provides the characterization of the family of indices satisfying the two basic axioms. Pro-rata and user-centric arise from the family by imposing an additional axiom. *Equal individual impact of similar users* for the former and *equal global impact of users* for the latter. Both axioms reflect different ways to address users and streams. The former focuses on streams while stating, roughly speaking, that all streams are equally important, independently of who has produced the streams. The latter focuses on users while stating, roughly speaking, that all users are equally important, independently of the streams of each user. Thus, we can see pro-rata and user-centric as two somewhat polar weighted indices. The first one "equalizes" streams, whereas the second one "equalizes" users.

We conclude this section complementing the axiomatic analysis from Theorem 1 upon formalizing two axioms that capture ideas that have recently received attention in the literature on the music industry. We shall see that, although Theorem 1 puts both the user-centric index and the pro-rata index on a similar foot, the following analysis (with respect to the new two axioms) favors the user-centric index.

Haampland et al. (2022) write the following: "such a payment system can be perceived to be quite unfair since individual users' payments are not aligned with their actual musical preferences." Similarly, Meyn et al. (2023) argue that users could "prefer that their payment only goes towards the content they use". Thus, we consider an axiom that says that given a set of users C and the set of artists A listened by those users, the amount received by the artists in A should be, at least, the amount paid by users in C. Formally,

Reasonable lower bound. For each $(N, M, t) \in \mathcal{P}$ and each $C \subset M$, let $L^C = \bigcup_{j \in C} L^j$. Then,

$$\sum_{i \in L^C} \frac{I_i (N, M, t)}{\sum_{i' \in N} I_{i'} (N, M, t)} m \geq |C| \, .$$

On another matter, Meyn et al. (2023) write the following: "some artists have asked their listeners to play their music silently while asleep to generate a larger share of remuneration". Dimont (2017) calls this phenomenon *click fraud* and discuss it giving some real examples. We first notice that it is not possible to know if some user is making click fraud or not (namely, if he/she is listening the song or not when playing it). Thus, it seems reasonable that if some song is played, the artists should be rewarded for it. Nevertheless, it should not be rewarded too much. For instance, it does not seem reasonable that if a user pays \$10 and decides to listen to an artist many times, then the amount received by this artist increases more than \$50. Otherwise, artists would have incentives to create fictitious users listening artist's songs all the day. We formalize this axiom by saying that if a user changes his/her streaming times, then the amount received by each artist could not change more that the subscription paid by the user. Formally,

Click-fraud-proofness. Let $(N, M, t), (N, M, t') \in \mathcal{P}$ and $j \in M$ be such that $t_{ij'} = t'_{ij'}$ for all $j' \in M \setminus \{j\}$ and $i \in N$. Then, for all $i \in N$,

$$\left| \frac{I_i(N, M, t)}{\sum\limits_{i' \in N} I_{i'}(N, M, t)} m - \frac{I_i(N, M, t')}{\sum\limits_{i' \in N} I_{i'}(N, M, t')} m \right| \leq 1.$$

The user-centric index satisfies the previous two axioms, whereas pro-rata satisfies none of them (see the appendix for the details).

4 A game-theoretical approach

A standard approach to solve resource allocation problems is through cooperative game theory. That is, given a resource allocation problem we associate a cooperative game, we compute a cooperative solution (for instance the core), and finally we compute the allocation induced by the cooperative solution in the original problem. We shall follow that approach in this section to solve our streaming problems.

A **cooperative game with transferable utility**, briefly a **TU game**, is a pair (N, v), where N denotes a set of agents and $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. The **core** of a cooperative game is defined as the set of feasible payoff vectors, for which no coalition can improve upon. Formally, for each game (N, v),

$$C(N, v) = \left\{ x = (x_i)_{i \in N} \colon \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subset N \right\}.$$

We now associate with each streaming problem (N, M, t) a *TU* game $\left(N, v_{(N,M,t)}\right)$ where the set of agents of the cooperative game is the set of artists. Given $S \subset N$ we define $v_{(N,M,t)}(S)$ as the amount paid by the users that have only listened to artists in S.[15] Formally,

$$v_{(N,M,t)}(S) = \left| \left\{ j \in M : L^j \subset S \right\} \right|.$$

When no confusion arises we write v instead of $v_{(N,M,t)}$. Note that this game is based on a pessimistic premise. To wit, it is reasonable to assume that if artists do not consider themselves to be well paid, they might leave the platform.[16] The issue is to estimate the revenues seceding artists might obtain after leaving the platform (and creating a new one). Our estimation is that just all users that listened only to the seceding artists will leave the platform to join the new one promoted by them. We acknowledge that this could be a quite pessimistic stance. For instance, consider a user whose 99% of streams are devoted to artists within the seceding group (S) and only 1% to other artists. It seems plausible to conjecture that this user would leave the platform, whereas our assumption on $v(S)$ does not consider it.

It is not difficult to show that v is a convex game, i.e., the incentives to join a coalition increase as the coalition grows (or, more formally, its characteristic function is supermodular). It is well known that the core

[15] The definition of $\left(N, v_{(N,M,t)}\right)$ is similar to the one used in the so-called museum pass problem (e.g., Ginsburg and Zang, 2003; Bergantiños and Moreno-Ternero, 2015).

[16] Kanye West, a highly popular artist, recently launched his own streaming service (e.g., Meyn et al., 2023).

of a convex game is quite large. The next result actually characterizes all the allocations within the core of this game. In words, they must satisfy the following: the amount paid by each user (which we have normalized to 1) is divided in any way among the artists listened by this user. Each artist then receives the sum (over all users) of the corresponding amounts. Formally, for each (N, M, t) we define the following set of allocations:

$$A(N, M, t) = \left\{ \begin{array}{c} x \in \mathbb{R}^N : x = \sum_{j \in M} x^j \text{ where for each } j \in M \\ x^j \in \mathbb{R}^N, \\ x_i^j = 0 \text{ for each } i \in N \backslash L^j, \\ x_i^j \geq 0 \text{ for each } i \in L^j \text{ and} \\ \sum_{i \in N} x_i^j = 1. \end{array} \right\}.$$

Theorem 2 *For each* $(N, M, t) \in \mathcal{P}$,

$$C(N, v) = A(N, M, t).$$

The proof of Theorem 2 can be found in the appendix.

We note that the rewards induced by the pro-rata index could be outside the core. To do so, consider for instance Example 1 and $S = \{1\}$. Then, $R_1^P(N, M, t) = \frac{20}{100} < 1 = v(S)$.

On the other hand, by definition, the user-centric index yields for each problem (N, M, t) an allocation that belongs to $A(N, M, t)$. Thus, by Theorem 2, the rewards induced by the user-centric index always belong to the core. More generally, we introduce the axiom stating that the rewards generated by an index should always lie within the core of the associated cooperative game. Formally,

Core selection. For each $(N, M, t) \in \mathcal{P}$, $R^I(N, M, t) \in C(N, v)$.

As the next result states, *core selection* actually characterizes the user-centric index, when combined with the axioms of *homogeneity* and *additivity* (already introduced in the previous section), provided we restrict to the following relevant subdomain of streaming problems.

More precisely, let $\mathcal{P}^*$ be the set of all problems (with at least three users) where no user has played content from all the artists. Namely,

$$\mathcal{P}^* = \left\{(N, M, t) \in \mathcal{P} : |M| \geq 3 \text{ and } L^j(N, M, t) \neq N \text{ for all } j \in M\right\}.$$

Theorem 3 *In the domain* $\mathcal{P}^*$*, an index satisfies homogeneity, additivity, and core selection if and only if it is the user-centric index.*

The proof of Theorem 3 can be found in the appendix. Note that $\mathcal{P}^*$ is not a very restrictive domain for real-life platforms.[17] Nevertheless, one could extend Theorem 3 to the full domain $\mathcal{P}$ upon simply adding an axiom of *independence of null artists*, stating that removing an artist with no streamings does not change the value of the index for the remaining artists.

[17] Obviously, all platforms have at least three users. And a user can play at most 89280 streams per month (assuming 30 seconds to qualify as a valid streaming, and the user is somehow playing 24/7 those shortest valid streamings), which is typically below the number of artists platforms have in their catalogue.

Based on the analysis in this section, we can safely state that the game-theoretical approach favors the user-centric index with respect to the pro-rata index.

5 A claims approach

In this section, we consider another (indirect) approach to solve streaming problems, based on claims problems. Claims problems refer to an amount of a homogeneous and infinitely divisible good (e.g., money) to be divided among a set of agents, who have claims on the good. This is certainly the case of the canonical and well-known bankruptcy problem.[18] Formally, a **bankruptcy problem** is a triple (N, c, E) where N is the set of agents, $c \in \mathbb{R}_+^N$ is the claims vector and $E \in \mathbb{R}_+$ is the amount to be divided. It is assumed that $\sum_{i \in N} c_i \geq E$. A rule is a function R assigning to each bankruptcy problem (N, c, E) a vector $R(N, c, E) \in \mathbb{R}^N$ such that for each $i \in N$, $0 \leq R_i(N, c, E) \leq c_i$ and $\sum_{i \in N} R_i(N, c, E) = E$. Some popular rules are the **proportional rule**, which yields awards proportionally to claims, and the **constrained equal awards rule**, which equalizes the amount received by each agent as much as possible. Formally, for each (N, c, E) and each $i \in N$,

$$P_i(N, c, E) = \frac{c_i}{\sum_{i' \in N} c_{i'}} E, \text{ and}$$

$$CEA_i(N, c, E) = \min\{\lambda, c_i\}$$

where λ satisfies $\sum_{i \in N} \min\{\lambda, c_i\} = E$.

We now associate with each streaming problem (N, M, t) a bankruptcy problem $\left(N, c(N, M, t), E_{(N,M,t)}\right)$ where $c_i(N, M, t) = T_i$, and $E(N, M, t) = m$. Notice that $R_i^P(N, M, t)$, the amount received by artist i under the pro-rata index, coincides with $P_i(N, c, m)$, the amount received by agent i in the associated bankruptcy problem under the proportional rule. Besides, $U_i(N, M, t)$, the amount received by artist i under the user-centric index, can not be computed through (N, T, m) because it depends on numbers (t_{ij}) that do not appear in (N, T, m). We now present an extension of bankruptcy problems, following Ju et al., (2007), that allows us to use the values t_{ij}.

A **claims problem** is a tuple (N, K, c, E) where N is a set of agents, K is a set of issues, $c = (c_{ij})_{i \in N, j \in K}$ where for all $i \in N$ and $j \in K$, $c_{ij} \geq 0$ denotes the characteristic of agent i on issue j, and $E \in \mathbb{R}_+$ is the amount of a homogeneous and infinitely divisible good to be divided. A rule is a function R assigning to each claims problem (N, K, c, E) a vector $R(N, K, c, E) \in \mathbb{R}^N$.

We now associate with each streaming problem (N, M, t) a claims problem $(N, K(N, M, t), c(N, M, t), E(N, M, t))$ where $K(N, M, t) = M$, $c(N, M, t) = t$, and $E(N, M, t) = m$.

Ju et al., (2007) characterize several families of rules.[19] One of the families is formally defined as follows.

For each claims problem (N, K, c, E) and each $j \in K$, let $c_{\cdot j} = (c_{ij})_{i \in N}$ and $C^j = \sum_{i \in N} c_{ij}$. A weight function is a function $\omega : \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}_+^K$, which assigns a probability distribution $\omega(x, y)$ (namely, $0 \leq w_j(x, y) \leq 1$ for

[18] Although this problem can be traced back to Aristotle and the Talmud, the seminal contribution is O'Neill (1982). Thomson (2019) is an excellent survey with an extensive treatment of the sizable literature emanating from that seminal contribution.

[19] The do so mostly thanks to the axiom of reallocation-proofness, which does not have a parallel in this paper.

all $j \in K$ and $\sum_{j \in K} w_j(x,y) = 1$). The **weighted proportional rule** associated to ω assigns, for each problem (N, K, c, E) and each $i \in N$ the amount

$$P_i^{\omega}(N, K, c, E) = \sum_{j \in K} \frac{c_{ij}}{C^j} \omega_j(\left(C^j\right)_{j \in K}, E)E.$$

In words, rule P^{ω} first applies the proportional rule to each single-dimensional sub-problem $(N, \{j\}, c_{.j}, E)$ and then it takes the weighted average of the solutions to the sub-problems according to the vector of weights $\omega_j(\left(C^j\right)_{j \in K}, E)$.

Calleja et al., (2005) introduce multi-issue allocation situations, which are a particular case of claims problems.[20] Bergantiños et al., (2010, 2011, 2018) consider two-stage rules for claims problems where in the first stage the endowment is divided among the issues and in the second stage the amount assigned to each issue is divided among the agents. The final amount received by each agent is the sum over all issues.

Formally, let ψ and ϕ be two bankruptcy rules. The **two-stage rule** $R^{\psi,\phi}$ is the claims rule obtained from the following two-stage procedure:

1. First stage. We consider the bankruptcy problem among the issues (K, c^K, E), where $c^K = (c_j^K)_{j \in K}$ and for each $j \in K$, $c_j^K = \sum_{i \in N} c_{ji}$. We compute $\psi(K, c^K, E)$.
2. Second stage. For each $j \in K$, we consider the bankruptcy problem $(N, \psi_j\left(K, E, c^K\right), c_{.j})$. We compute $\phi(N, c_{.j}, \psi_j\left(K, c^K, E\right))$.

Thus, for each $i \in N$,

$$R_i^{\psi,\phi}(N, K, C, E) = \sum_{j \in K} \phi_i\left(N, c_{.j}, \psi_j\left(K, c^K, E\right)\right).$$

Moreno-Ternero (2009) and Bergantiños et al., (2010) study the two-stage rule where the proportional rule is used in both stages (namely, $\psi = \phi = P$). Bergantiños et al. (2011) study the two-stage rule where the constrained equal awards rule is used in both stages (namely, $\psi = \phi = CEA$). Bergantiños et al. (2018) study the two-stage rule where the constrained equal awards rule is used in the first stage and the proportional rule in the second stage (namely, $\psi = CEA$ and $\phi = P$). But all those papers take the axiomatic approach inspired on the literature of bankruptcy problems, whch is unrelated to the axiomatic study of this paper.

We obtain the following relationships between the pro-rata index and the user-centric index and some of the rules from the literature on claims problems.

Theorem 4 *Let (N, M, t) be a streaming problem and (N, K, c, E) be the associated claims problem. Then,*

(a) There exist weight functions ω^P and ω^U such that $R^P(N, M, t) = P^{\omega^P}(N, K, c, E)$ and $R^U(N, M, t) = P^{\omega^U}(N, K, c, E)$.

(b) $R^P(N, M, t) = R^{P,P}(N, K, c, E)$.

(c) $U(N, M, t) = R^{CEA,P}(N, K, c, E)$.

[20] Actually the tuple (N, K, c, E) is defined as in claims problems but some additional constraints on c, E and the definition of a rule are added. As our results are not affected by that, we avoit the details.

The proof of Theorem 4 can be found in the appendix.

From Theorem 4 (a) we obtain that the pro-rata index and the user-centric index can be rationalized as weighted proportional rules. We have seen in Theorem 1 that they are weighted indices. Thus, one might naturally conjecture whether there is a bijection between weighted proportional rules (for claims problems) and weighted indices (for streaming problems). The answer is not. We can indeed find weighted proportional rules that can not be obtained through a weighted index and weighted indices such that the induced rule to allocate awards in streaming problems is not a weighted proportional rule.

Theorem 4 (b) and 4 (c) allow us to consider both indices from another perspective. As the statements indicate, the allcation rules they induce (for streaming problems) can actually be described as two-stage (bankruptcy) rules where we first decide the importance of each user an then the importance of each artist for each user, which is computed as the sum over all users. The pro-rata and user-centric indices measure the importance of each artist in different ways. For the latter, all users have the same importance, whereas for the former the importance of each user is proportional to the user's streams. They then measure the importance of each artist for each user in the same way; namely, proportionally to the artists' streams.

To conclude with this section, we reiterate that Theorem 4 states that both indices could be rationalized as a combination of two well-known rules from the literature on bankruptcy problems. Besides, both of them can also be seen as members of the same family of weighted proportional rules. Thus, we can conclude that (in contrast with the previous sections) the analysis in this section does not favor one of the indices over the other.

6 Conclusion

We have analyzed in this paper the problem of sharing the revenues raised from subscription fees to music platforms among participating artists. Our analysis has highlighted two central methods (pro-rata and user-centric) which can actually be seen as focal (and somewhat polar) members of a family of methods which evaluate artists by the weighted aggregation of users' streaming choices. The weight assigned to each user might actually depend on the user herself and her whole streaming profile. We therefore provide a solid common ground for both methods, in the form of the characterization result for the whole family. We, nevertheless, provide additional (normative, as well as game-theoretical) arguments to favor the user-centric method with respect to the pro-rata method. To wit, we show that the former satisfies two natural and appealing axioms (reasonable lower bound and click-fraud proofness) that the second violates. They are somewhat connected to a feature the second exhibits (whereas the first does not); namely, cross-subsidization between high- and low-streaming-volume users. Furthermore, the former satisfies core-selection, while the latter does not (which implies that it does not guarantee allocations preventing incentives for artist to leave the platform)

To complement the above analysis, we simply stress that two aspects are relevant when it comes to measuring the importance of artists (within a platform). On the one hand, the users playing content from the artist. On the other hand, the streaming times the artist achieved. Although both aspects are related, they may differ largely. For instance, two artists may have been listened by the same number of users but with very different streaming times. We believe the pro-rata index manages well the streaming times but not the number of users.

The user-centric index manages well both the streaming times and the number of users.

Our analysis should contribute to the debate between the pro-rata and user-centric methods in the music industry. Nevertheless, the discussion in that industry nowadays goes beyond the debate between those two methods. For instance, the French streaming service Deezer claims to be pioneer in fair payments to artist, "being a main advocate for a re-evaluation of music streaming's economic model". In March 2023, Deezer announced an initiative with Universal Music Group, the world leader in music-based entertainment, to explore new streaming models that better align the interests of artists, fans and streaming services. Using deep data analysis, this partnership aims to improve the fairness of the current streaming model in various ways, whether by helping artists monetize their music better or by eliminating issues within the current system. This initiative will not prioritize just the most-streamed artists on the platform, but will level the playing field for artists at every stage of their career and benefit the wider music community as a whole.[21] We believe that some of these goals might be achieved with other members of the more general family of weigthed indices we characterize in this paper. As we mentioned above, weigthed indices are precisely constructed on the premise that each artist is assigned the weighted aggregation of streamings, accross users, with the weight depending on the user and her streaming profile.

7 Appendix

Proof of Theorem 1

(a) We first prove that each weighted index satisfies the two axioms.

For each ω, I^{ω} satisfies *homogeneity*. Let $(N,M,t)\in\mathcal{P}$, $i,i'\in N$, and $\lambda\in\mathbb{N}_{+}$ such that $t_{ij}=\lambda t_{i'j}$ for all $j\in M$,

$$I_i^{\omega}(N,M,t)=\sum_{j\in M}\omega(j,t_{\cdot j})\,t_{ij}=\sum_{j\in M}\omega(j,t_{\cdot j})\,\lambda t_{i'j}=\lambda I_{i'}^{\omega}(N,M,t).$$

For each ω, I^{ω} satisfies *additivity*. Let $(N,M^1,t^1),(N,M^2,t^2),(N,M,t)\in\mathcal{P}$ such that $M=M^1\cup M^2$, $t_{ij}=t_{ij}^1$ when $j\in M^1$ and $t_{ij}=t_{ij}^2$ when $j\in M^2$. For each $i\in N$,

$$\begin{aligned}I_i^{\omega}(N,M^1,t^1)+I_i^{\omega}(N,M^2,t^2) &= \sum_{j\in M^1}\omega(j,t_{\cdot j}^1)\,t_{ij}^1+\sum_{j\in M^2}\omega(j,t_{\cdot j}^2)\,t_{ij}^2\\ &= \sum_{j\in M}\omega(j,t_{\cdot j})\,t_{ij}=I_i^{\omega}(N,M,t).\end{aligned}$$

Conversely, we now prove that if an index I satisfies the two axioms, then there exists a weight system for streams ω such that $I=I^{\omega}$.

Let $(N,M,t)\in\mathcal{P}$. By *additivity*, for each $i\in N$,

$$I_i(N,M,t)=\sum_{j\in M}I_i(N,\{j\},t_{\cdot j}).$$

Let $j\in M$. By *homogeneity*, for each $i\in N$ such that $t_{ij}=0$, $I_i(N,\{j\},t_{\cdot j})=0$.

[21] See https://www.deezer-blog.com/how-much-does-deezer-pay-artists/

Recall the basic assumption in our model that $L^j(N,\{j\},t_{.j}) \neq \varnothing$. Then, let $i \in L^j(N,\{j\},t_{.j})$. By *homogeneity*, for each $i' \in L^j(N,\{j\},t_{.j})$,

$$I_{i'}(N,\{j\},t_{.j}) = \frac{t_{i'j}}{t_{ij}} I_i(N,\{j\},t_{.j}).$$

Then,

$$\begin{aligned}\sum_{i' \in N} I_{i'}(N,\{j\},t_{.j}) &= \sum_{i' \in L^j(N,\{j\},t)} I_{i'}(N,\{j\},t_{.j}) \\ &= \sum_{i' \in L^j(N,\{j\},t)} \frac{t_{i'j}}{t_{ij}} I_i(N,\{j\},t_{.j}) \\ &= \frac{T^j(N,\{j\},t_{.j})}{t_{ij}} I_i(N,\{j\},t_{.j}).\end{aligned}$$

Hence,

$$I_i(N,\{j\},t_{.j}) = \frac{\sum_{i' \in N} I_{i'}(N,\{j\},t_{.j})}{T^j(N,\{j\},t_{.j})} t_{ij}.$$

Taking $\omega(j,t_{.j}) = \frac{\sum_{i' \in N} I_{i'}(N,\{j\},t_{.j})}{T^j(N,\{j\},t_{.j})}$ we have that

$$I_i(N,M,t) = \sum_{j \in M} I_i(N,\{j\},t_{.j}) = \sum_{j \in M} \omega(j,t_{.j}) t_{ij} = I_i^{\omega}(N,M,t).$$

(b) As the pro-rata index is a weighted index, we know it satisfies *homogeneity* and *additivity*. As for *equal individual impact of similar users*, let $(N,M,t) \in \mathcal{P}$, $i \in N$ and $j,j' \in M$ such that $t_{ij} = t_{ij'}$. Then,

$$\begin{aligned}P_i\left(N, M\backslash\{j\}, t^{-j}\right) &= T_i\left(N,M,t^{-j}\right) - t_{ij} \\ &= T_i\left(N,M,t^{-j'}\right) - t_{ij'} \\ &= P_i\left(N, M\backslash\{j'\}, t^{-j'}\right).\end{aligned}$$

Let now I be an index satisfying the three axioms in the second statement. By the proof of the previous statement, we know that $I = I^{\omega}$ for some weight system ω.

We now prove that $\omega(j,t_{.j})$ does not depend on j and $t_{.j}$. Let $j \in M$ and $i \in L^j(N,\{j\},t_{.j})$. For each $x > 0$ consider the problem $(N,\{j'\},t^x)$ such that $t_{ij'} = x$ for all $i \in N$. By *homogeneity*, for each pair $i,i' \in N$, $I_i(N,\{j'\},t^x) = I_{i'}(N,\{j'\},t^x)$. Let $(N,M',t') \in \mathcal{P}$ be such that $M' = \{j,j'\}$ and for all $i' \in N$, $t'_{i'j} = t_{i'j}$ and $t'_{i'j'} = t^{t_{ij}}_{i'j'}$.

By *equal individual impact of similar users*,

$$I_i\left(N, M'\backslash\{j\}, t^{-j}\right) = I_i\left(N, M'\backslash\{j'\}, t^{-j'}\right).$$

Then,

$$\begin{aligned}\omega(j,t_{.j}) t_{ij} &= I_i(N,\{j\},t_{.j}) = I_i\left(N, M'\backslash\{j'\}, t^{-j'}\right) \\ &= I_i\left(N, M'\backslash\{j\}, t^{-j}\right) = I_i\left(N,\{j'\},t^{t_{ij}}\right) \\ &= \omega\left(j', t^{t_{ij}}\right) t_{ij}.\end{aligned}$$

Hence, $\omega(j, t_{.j}) = \omega(j', t^{t_{ij}})$.

Let $t^* \in \mathbb{N}_+^N$ be such that $t^*_{ij} = t_{ij}$ and $t^*_{i'j} = 1$ when $i' \neq j$. Similarly to $\omega(j, t_{.j})$, we can argue that $\omega(j, t^*) = \omega(j', t^{t_{ij}})$.

Take $(N, \{j\}, t^*) \in \mathcal{P}$ and $i' \neq i$. Similarly to $\omega(j, t_{.j})$ with i' instead of i we can argue that $\omega(j, t^*) = \omega\left(j', t^1\right)$. Thus, for each $j, j' \in M$ and each $t_{.j}$ we have that $\omega(j, t_{.j}) = \omega\left(j', t^1\right)$. Then, $\omega(j, t_{.j})$ does not depend on j and $t_{.j}$ and we can define

$$\lambda = \omega(j, t_{.j}).$$

Now, for each (N, M, t) and each $i \in N$,

$$I_i(N, M, t) = I_i^{\omega}(N, M, t) = \sum_{j \in M} \lambda t_{ij} = \lambda T_i = \lambda P_i(N, M, t).$$

Then, I is a (positive) linear transformation of P and hence it coincides with P (recall that we define indices up to a positive linear transformation, because the resulting ones generate the same allocation of rewards among artists).

(3) As the user-centric index is a weighted index, we know it satisfies *homogeneity* and *additivity*. As for *equal global impact of users*, let $(N, M, t) \in \mathcal{P}$ and $j \in M$,

$$\sum_{i \in N} U_i\left(N, M \backslash \{j\}, t^{-j}\right) = \sum_{i \in N} \sum_{k \in M \backslash \{j\}} \frac{t_{ik}}{T^k} = \sum_{k \in M \backslash \{j\}} \sum_{i \in N} \frac{t_{ik}}{T^k} = \sum_{k \in M \backslash \{j\}} 1 = m - 1.$$

Thus, $\sum_{i \in N} U_i\left(N, M \backslash \{j\}, t^{-j}\right)$ does not depend on $j \in M$ and hence the user-centric index satisfies *equal global impact of users*.

Let now I be an index satisfying the axioms in the third statement. By the proof of the first statement, for each $(N, M, t) \in \mathcal{P}$ and each $i \in N$,

$$I_i(N, M, t) = \sum_{j \in M} \omega(j, t_{.j}) t_{ij},$$

where

$$\omega(j, t_{.j}) = \frac{\sum_{i' \in N} I_{i'}(N, \{j\}, t_{.j})}{T^j(N, \{j\}, t_{.j})}$$

Let $j, j' \in M$ and $M^* = \{j, j'\}$. By *equal global impact of users*,

$$\sum_{i \in N} I_i(N, M^* \backslash \{j\}, t_{.j'}) = \sum_{i \in N} I_i(N, M^* \backslash \{j'\}, t_{.j}).$$

Thus,

$$\sum_{i \in N} I_i(N, \{j'\}, t_{.j'}) = \sum_{i \in N} I_i(N, \{j\}, t_{.j}).$$

Then, $\sum_{i \in N} I_i(N, \{j\}, t_{.j})$ does not depend on j and $t_{.j}$ and we can define

$$\lambda = \sum_{i \in N} I_i(N, \{j\}, t_{.j}).$$

As $T^j(N, \{j\}, t_{.j}) = T^j(N, M, t)$, it follows that

$$I_i(N, M, t) = \sum_{j \in M} \frac{\lambda}{T^j(N, M, t)} t_{ij} = \lambda U_i(N, M, t).$$

Then, I is a (positive) linear transformation of U and hence it coincides with U.

Independence of the axioms of Theorem 1

We now show that all the axioms used in the previous characterizations are independent.

(a) Let I^1 be the uniform index that assigns to each artist a constant number. Namely, for each (N, M, t) and each $i \in N$, $I^1_i(N, M, t) = 1$. I^1 satisfies *additivity* but not *homogeneity*.

Let I^2 be defined as follows. For each (N, M, t) and each $i \in N$,

$$I^2_i(N, M, t) = \sum_{j=1}^{m} \frac{t_{ij} + T_i}{T^j + \sum_{i' \in N} T_{i'}}.$$

I^2 satisfies *homogeneity* but not *additivity*.

(b) Let I^3 be defined as follows. For each (N, M, t) and each $i \in N$,

$$I^3_i(N, M, t) = \sum_{j=1}^{m} t^2_{ij}.$$

I^3 satisfies *additivity* and *equal individual impact of similar users* but not *homogeneity*.

I^2, defined above, satisfies *homogeneity* and *equal individual impact of similar users*, but not *additivity*.

The user-centric index satisfies *homogeneity* and *additivity* but not *equal individual impact of similar users*.

(c) I^1, defined above, satisfies *additivity* and *equal global impact of users*, but not *homogeneity*.

Let I^4 be defined as follows. For each (N, M, t) and each $i \in N$,

$$I^4_i(N, M, t) = \frac{T_i}{\sum_{i' \in N} T_{i'}} m.$$

I^4 satisfies *homogeneity* and *equal global impact of users*, but not *additivity*.

The pro-rata index satisfies *homogeneity* and *additivity*, but not *equal global impact of users*.

Reasonable lower bound and click-fraud-proofness

The user-centric index satisfies reasonable lower bound. Let $(N, M, t) \in \mathcal{P}$, $C \subset M$ and $L^C = \bigcup_{j \in C} L^j$. Then,

$$\begin{aligned}
\sum_{i \in L^C} \frac{U_i(N, M, t)}{\sum_{i' \in N} U_{i'}(N, M, t)} m &= \sum_{i \in L^C} U_i(N, M, t) = \sum_{i \in L^C} \sum_{j \in M} \frac{t_{ij}}{T^j} \\
&\geq \sum_{i \in L^C} \sum_{j \in C} \frac{t_{ij}}{T^j} = \sum_{j \in C} \sum_{i \in L^C} \frac{t_{ij}}{T^j} \\
&= \sum_{j \in C} 1 = |C|.
\end{aligned}$$

The pro-rata index does not satisfy it. To show that, consider Example 1 and let $C = \{a\}$. Then $L^a = \{1\}$ but

$$\frac{P_1(N, M, t)}{\sum_{i' \in N} P_{i'}(N, M, t)} 2 = \frac{20}{100} < 1.$$

The user-centric index satisfies click-fraud-proofness. To show this, let (N, M, t), (N, M, t'), and $j \in M$ such that $t_{ij'} = t'_{ij'}$ for all $j' \in M \backslash \{j\}$. Then, for all $i \in N$,

$$\begin{aligned} \left| \frac{U_i(N,M,t)}{\sum\limits_{i' \in N} U_{i'}(N,M,t)} m - \frac{U_i(N,M,t')}{\sum\limits_{i' \in N} U_{i'}(N,M,t')} m \right| &= |U_i(N,M,t) - U_i(N,M,t')| \\ &= \left| \frac{t_{ij}}{T^j(N,M,t)} - \frac{t'_{ij}}{T^j(N,M,t')} \right| \\ &\leq 1, \end{aligned}$$

where the last inequality is true because $\frac{t_{ij}}{T^j(N,M,t)}, \frac{t'_{ij}}{T^j(N,M,t')} \in [0,1]$.

The pro-rata index, however, does not satisfy it. To show that, consider Example 1. If we let

$$t' = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix},$$

we obtain

$$\left| \frac{P_2(N,M,t)}{\sum\limits_{i' \in N} P_{i'}(N,M,t)} m - \frac{P_2\left(N,M',t'\right)}{\sum\limits_{i' \in N} P_{i'}(N,M',t')} m \right| = \left| \frac{90}{100} 2 - \frac{2}{12} 2 \right| = \frac{22}{15} > 1.$$

Proof of Theorem 2

We prove Theorem 2 using a classical result from the literature on cooperative game theory. We first introduce some notation. We then explain the classical result. Finally, we apply it to our game (N,v).

The unanimity game u_R associated with the nonempty coalition $R \subset N$ is defined as $u_R(S) = 1$ when $R \subset S$ and $u_R(S) = 0$ otherwise.

Harsanyi (1959) proved that for every TU game (N,v), there exist unique weights $\Delta_v(R) \in \mathbb{R}$ such that $v = \sum\limits_{\varnothing \neq R \subset N} \Delta_v(R) u_R$. The weights $\Delta_v(R)$ are called the (Harsanyi) dividends of v.

Given a TU game v, the Harsanyi set $H(N,v)$ (Vasil'ev 1978, 1981) is defined as the set of allocations obtained by distributing the dividend of any coalition R in any way among the agents in R. We now introduce it formally.[22]

We say that $p = \left(p^R\right)_{\varnothing \neq R \subset N}$ is a sharing system if for each $\varnothing \neq R \subset N$ we have that $p^R \in \mathbb{R}^N$, $p_i^R \geq 0$ for all $i \in R$, $p_i = 0$ for all $i \in N \backslash R$, and $\sum\limits_{i \in N} p_i^R = 1$. Each sharing system p induces the allocation y^p where for each $i \in N$,

$$y_i^p = \sum_{i \in R} \Delta_v(R) p_i^R.$$

The Harsanyi set is defined as

$$H(N,v) = \{y^p : p \text{ is a sharing system}\}.$$

Vasil'ev (1981) proved that if all dividends of (N,v) are nonnegative, $C(N,v) = H(N,v)$.

It is straightforward to check that given a streaming problem (N,M,t),

$$v = \sum_{\varnothing \neq R \subset N} \left| j \in M : L^j = R \right| u_R.$$

[22] The presentation we consider here follows the one in van den Brink et al., (2014).

Then, for each $\varnothing \neq R \subset N$, $\Delta_v(R) = \left|j \in M : L^j = R\right| \geq 0$. Thus, for our result, it suffices to prove that $H(N,v) = A(N,M,t)$.

We first prove that $H(N,v) \subset A(N,M,t)$. Let p be a sharing system. We must prove that $y^p \in A(N,M,t)$. We define $\left(x^j\right)_{j\in M}$ and $x = \sum_{j\in M} x^j \in A(N,M,t)$ as follows. Given $j \in M$ we take $x^j = p^{L^j}$. Because of the definition of a sharing system we have that x^j satisfies the condition of $A(N,M,t)$. Besides, for each $i \in N$

$$x_i = \sum_{j\in M} x_i^j = \sum_{j\in M: i\in L^j} p_i^{L^j} = \sum_{i\in R\subset N} \left|j \in M : L^j = R\right| p_i^R = y_i^p.$$

We now prove that $A(N,M,t) \subset H(N,v)$. Let $\left(x^j\right)_{j\in M}$ and $x = \sum_{j\in M} x^j \in A(N,M,t)$. We define p as follows. For each $R \subset N$ and $i \in N$,

$$p_i^R = \begin{cases} \frac{\sum_{j\in M, L^j=R} x_i^j}{|j\in M, L^j=R|} & \text{when } \left|j \in M, L^j = R\right| > 0 \\ \frac{1}{|N|} & \text{when } \left|j \in M, L^j = R\right| = 0. \end{cases}$$

We prove that p is a sharing system. Let $R \subset N$. Since $x_i^j \geq 0$ for all $i \in N$ and $j \in M$, $p_i^R \geq 0$ when $i \in R$. Since $x_i^j = 0$ when $i \in N \backslash L^j$, $p_i^R = 0$ when $i \in N\backslash R$. If $\left|j \in M, L^j = R\right| = 0$, obviously $\sum_{i\in N} p_i^R = 1$. Assume that $\left|j \in M, L^j = R\right| > 0$. Now,

$$\sum_{i\in N} p_i^R = \sum_{i\in N} \frac{\sum_{j\in M, L^j=R} x_i^j}{|j \in M, L^j = R|} = \sum_{j\in M, L^j=R} \frac{\sum_{i\in N} x_i^j}{|j \in M, L^j = R|} = \sum_{j\in M, L^j=R} \frac{1}{|j \in M, L^j = R|} = 1.$$

Besides, for each $i \in N$,

$$\begin{aligned} y_i^p &= \sum_{i\in R\subset N} \left|j \in M : L^j = R\right| p_i^R \\ &= \sum_{i\in R\subset N} \left|j \in M : L^j = R\right| \frac{\sum_{j\in M, L^j=R} x_i^j}{|j \in M, L^j = R|} \\ &= \sum_{i\in R\subset N} \sum_{j\in M, L^j=R} x_i^j = \sum_{j\in M} \sum_{i\in L_j} x_i^j = \sum_{j\in M} x_i^j \\ &= x_i. \end{aligned}$$

Proof of Theorem 3

We have already seen that the user-centric index satisfies *homogeneity*, *additivity*, and *core selection*.

Conversely, let I be an index satisfying the axioms. Let $(N,M,t) \in \mathcal{P}^*$ and $j \in M$. We consider the problem $(N, \{j\}, t_{.j}) \in \mathcal{P}^*$. By *core selection*, taking $S = L^j(N, \{j\}, t_{.j}) \neq N$, we have

$$\sum_{i\in S} \frac{I_i(N, \{j\}, t_{.j})}{\sum_{i'\in N} I_{i'}(N, \{j\}, t_{.j})} \geq v_{(N,\{j\},t_{.j})}(S) = 1,$$

which implies

$$\sum_{i\in L^j(N,\{j\},x)} I_i(N, \{j\}, t_{.j}) \geq \sum_{i'\in N} I_{i'}(N, \{j\}, t_{.j}).$$

Thus,

$$\sum_{i\in N\backslash L^j(N,\{j\},x)} I_i(N, \{j\}, t_{.j}) \leq 0.$$

And, therefore,

$$I_i(N,\{j\},t_{.j})=0 \text{ for each } i\in N\backslash L^j(N,\{j\},t_{.j}). \tag{1}$$

Now, let $i,i'\in L^j(N,\{j\},t_{.j})$. By *homogeneity*,

$$I_{i'}(N,\{j\},t_{.j})=\frac{t_{i'j}}{t_{ij}}I_i(N,\{j\},t_{.j}).$$

Then, for each $i\in L^j(N,\{j\},t_{.j})$,

$$\sum_{i'\in N}I_{i'}(N,\{j\},t_{.j})=\frac{\sum\limits_{i'\in N}t_{i'j}}{t_{ij}}I_i(N,\{j\},t_{.j}).$$

Hence,

$$I_i(N,\{j\},t_{.j})=\left[\sum_{i'\in N}I_{i'}(N,\{j\},t_{.j})\right]\frac{t_{ij}}{\sum\limits_{i'\in N}t_{i'j}}=\left[\sum_{i'\in N}I_{i'}(N,\{j\},t_{.j})\right]U_i(N,\{j\},t_{.j}).$$

We now prove the following claim.

Claim. For each pair $j^1,j^2\in M$, such that $j^1\neq j^2$ and $L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)\cap L^{j^2}\left(N,\{j^2\},t_{.j^2}\right)=\varnothing$ we have

$$\sum_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)=\sum_{i'\in N}I_{i'}\left(N,\left\{j^2\right\},t_{.j^2}\right).$$

Suppose that the claim is false. Then, without loss of generality, assume that

$$\sum_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)<\sum_{i'\in N}I_{i'}\left(N,\left\{j^2\right\},t_{.j^2}\right).$$

Let $\left(N,\left\{j^1,j^2\right\},\hat{t}\right)\in\mathcal{P}^*$ where $\hat{t}$ denotes the restriction of t to $\left\{j^1,j^2\right\}$. By *core selection*, taking $S=L^{j^1}\left(N,\left\{j^1\right\},t_{.j^1}\right)$,

$$1=v_{\left(N,\{j^1,j^2\},\hat{t}\right)}(S)\leq\sum_{i\in S}\frac{I_i\left(N,\left\{j^1,j^2\right\},\hat{t}\right)}{\sum\limits_{i'\in N}I_{i'}\left(N,\left\{j^1,j^2\right\},\hat{t}\right)}2$$

By *additivity*,

$$\begin{aligned}\sum_{i'\in N}I_{i'}\left(N,\left\{j^1,j^2\right\},\hat{t}\right)&=\sum_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)+\sum_{i'\in N}I_{i'}\left(N,\left\{j^2\right\},t_{.j^2}\right)\text{ and}\\ \sum_{i\in L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)}I_i\left(N,\left\{j^1,j^2\right\},\hat{t}\right)&=\sum_{i\in L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)}I_i\left(N,\left\{j^1\right\},t_{.j^1}\right)+\sum_{i\in L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)}I_i\left(N,\left\{j^2\right\},t_{.j^2}\right).\end{aligned}$$

By (1),

$$\begin{aligned}\sum_{i\in L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)}I_i\left(N,\left\{j^1\right\},t_{.j^1}\right)&=\sum_{i\in N}I_i\left(N,\left\{j^1\right\},t_{.j^1}\right)\text{ and}\\ \sum_{i\in L^{j^1}\left(N,\{j^1\},t_{.j^1}\right)}I_i\left(N,\left\{j^2\right\},t_{.j^2}\right)&=0.\end{aligned}$$

Then,

$$\begin{aligned}1&\leq\frac{\sum\limits_{i\in N}I_i\left(N,\left\{j^1\right\},t_{.j^1}\right)}{\sum\limits_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)+\sum\limits_{i'\in N}I_{i'}\left(N,\left\{j^2\right\},t_{.j^2}\right)}2\\ &<\frac{\sum\limits_{i\in N}I_i\left(N,\left\{j^1\right\},t_{.j^1}\right)}{\sum\limits_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)+\sum\limits_{i'\in N}I_{i'}\left(N,\left\{j^1\right\},t_{.j^1}\right)}2=1,\end{aligned}$$

which is a contradiction. Hence, the claim holds.

We now prove that, for each pair $j^1, j^2 \in M$, with $j^1 \neq j^2$

$$\sum_{i' \in N} I_{i'}\left(N, \left\{j^1\right\}, t_{.j^1}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^2\right\}, t_{.j^2}\right).$$

By the claim, the result holds when $L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \cap L^{j^2}\left(N, \left\{j^2\right\}, t_{.j^2}\right) = \varnothing$. Assume then otherwise. We consider two cases.

1. $L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \supset L^{j^2}\left(N, \left\{j^2\right\}, t_{.j^2}\right)$.[23]

   As $\left(N, \left\{j^1\right\}, t_{.j^1}\right) \in \mathcal{P}^*$, it follows that $S = N \setminus L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \neq \varnothing$. Let $j^3 \in M \setminus \left\{j^1, j^2\right\}$ and $t^S_{.j^3}$ be such that

   $$t^S_{ij^3} = \begin{cases} 1 & \text{for each } i \in S, \\ 0 & \text{for each } i \in N \setminus S. \end{cases}$$

   Then, by the Claim above,

   $$\sum_{i' \in N} I_{i'}\left(N, \left\{j^1\right\}, t_{.j^1}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^3\right\}, t^S_{.j^3}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^2\right\}, t_{.j^2}\right).$$

2. $S = L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \setminus L^{j^2}\left(N, \left\{j^2\right\}, t_{.j^2}\right) \neq \varnothing$ and $S' = L^{j^2}\left(N, \left\{j^2\right\}, t_{.j^2}\right) \setminus L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \neq \varnothing$. Let $t^S_{.j^1}$ and $t^{S'}_{.j^2}$ be defined as in Case 1. Then, by the Claim above,

   $$\sum_{i' \in N} I_{i'}\left(N, \left\{j^1\right\}, t_{.j^1}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^2\right\}, t^{S'}_{.j^2}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^1\right\}, t^S_{.j^1}\right) = \sum_{i' \in N} I_{i'}\left(N, \left\{j^2\right\}, t_{.j^2}\right).$$

To conclude the proof, we define $p = \sum_{i' \in N} I_{i'}\left(N, \{j\}, t_{.j}\right)$. Based on the above, p is well defined because it neither depends on j, nor on $t_{.j}$. Then,

$$I_i\left(N, \{j\}, t_{.j}\right) = pU_i\left(N, \{j\}, t_{.j}\right).$$

By *additivity*,

$$I_i\left(N, M, t\right) = \sum_{j \in M} I_i\left(N, \{j\}, t_{.j}\right) = \sum_{j \in M} pU_i\left(N, \{j\}, t_{.j}\right) = pU_i\left(N, M, t\right).$$

Thus, I is also the user-centric index, as desired.

Independence of the axioms of Theorem 3

We now show that all the axioms used in the previous characterization are independent.

Let I^5 be defined as follows. For each (N, M, t) and each $i \in N$,

$$I^5_i\left(N, M, t\right) = \sum_{j \in M : i \in L^j(N, \{j\}, t_{.j})} \frac{1}{\left|L^j\left(N, \{j\}, t_{.j}\right)\right|}.$$

I^5 satisfies all axioms but *homogeneity*.

I^2, defined above, satisfies all axioms but *additivity*.

The pro-rata satisfies all axioms but *core selection*.

[23] The case $L^{j^1}\left(N, \left\{j^1\right\}, t_{.j^1}\right) \subset L^{j^2}\left(N, \left\{j^2\right\}, t_{.j^2}\right)$ is similar and, thus, we omit it.

Proof of Theorem 4

(a) We consider the weight function ω^P given by

$$\omega_j^P\left(\left(C^{j'}\right)_{j'\in K}, E\right) = \frac{C^j}{\sum\limits_{j'\in K} C^{j'}}.$$

For each $i \in N$,

$$P_i^{\omega^P}(N,K,c,E) = \sum_{j\in M} \frac{t_{ij}}{T^j}\frac{T^j}{\sum\limits_{j'\in K} T^{j'}} m = \frac{\sum\limits_{j\in M} t_{ij}}{\sum\limits_{j'\in K} T^{j'}} m = \frac{T_i}{\sum\limits_{i'\in N} T_{i'}} m = R_i^P(N,M,t).$$

Thus, R^P is the weighted proportional rule associated to ω^P.

We now take the weight function ω^U given by

$$\omega_j^U\left(\left(C^{j'}\right)_{j'\in K}, E\right) = \frac{1}{|K|}.$$

For each $i \in N$,

$$P_i^{\omega^U}(N,K,c,E) = \sum_{j\in M} \frac{t_{ij}}{T^j}\frac{1}{m} m = \sum_{j\in M} \frac{t_{ij}}{T^j} = U_i(N,M,t).$$

Thus U is the weighted proportional rule associated to ω^U.

(b) Given $i \in N$, we compute $R_i^{P,P}(N,K,c,E)$

First stage. For each $j \in K$,

$$P_j(K,c^K,E) = \frac{T^j}{\sum\limits_{j'\in K} T^{j'}} m.$$

Second stage.

$$P_i(N,c_{\cdot j},P_j\left(K,c^K,E\right)) = \frac{t_{ij}}{T^j}\frac{T^j}{\sum\limits_{j'\in M} T^{j'}} m = \frac{t_{ij}}{\sum\limits_{j'\in M} T^{j'}} m.$$

Thus, for each $i \in N$,

$$\begin{aligned} R_i^{P,P}(R,N,E,C) &= \sum_{j\in K} P_i(N,c_{\cdot j},P_j\left(K,c^K,E\right)) \\ &= \sum_{j\in M} \frac{t_{ij}}{\sum\limits_{j'\in M} T^{j'}} m = \frac{T_i}{\sum\limits_{i'\in N} T_{i'}} m = R_i^P(N,M,t). \end{aligned}$$

(c) Given $i \in N$, we compute $R_i^{CEA,P}(N,K,c,E)$.

First stage. For each $j \in K$, $CEA_j(K,c^K,E) = \min\left\{\lambda, T^j\right\}$ where $\sum\limits_{j\in K} \min\left\{\lambda, T^j\right\} = E$. It is straightforward to check that $\lambda = 1$. Hence,

$$CEA_j(K,c^K,E) = 1.$$

Second stage.

$$P_i(N,c_{\cdot j},CEA_j(K,c^K,E)) = \frac{t_{ij}}{T^j}.$$

Thus, for each $i \in N$,

$$R_i^{CEA,P}(R,N,E,C) = \sum_{j\in K} P_i(N, c_{.j}, CEA_j(K, c^K, E))) = \sum_{j\in M} \frac{c_{ij}}{T^j} = U_i(N,M,t).$$

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