On tuna fishing games[1]

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Abstract

Bergantiños et al. (2023) introduce cooperative tuna fishing games. We prove that the core of this game is not empty. We give an explicit formula for the τ-value. We also study the distance game associated with tuna fishing games and prove that it is a generalized big boss game. In most practical cases, the number of vessels is two or three. In these cases, we provide an explicit formula for the nucleolus. Moreover, the core and the core cover coincide.

Keywords: tuna fishing games; core; τ-value; nucleolus.

1. Introduction

Cooperative game theory has been used to analyze practical situations where a cost or benefit must be shared among heterogeneous agents. Classical examples include cost allocation problems (Tijs and Driessen, 1986), bankruptcy problems (Curiel et al., 1987), sequencing games (Curiel et al., 1989), traveling salesman games (Potters et al., 1992), telecommunications problems (van den Nouweland et al., 1996), and minimum cost spanning tree problems (Norde et al., 2004). Recently, new examples have emerged as broadcasting problems (Bergantiños and Moreno-Ternero, 2023), location problems (Navarro-Ramos, 2022; Bergantiños and Navarro-Ramos, 2023), or streaming problems (Schlicher et al., 2024, Gonçalves-Dosantos et al., 2025; and Bergantiños and Moreno-Ternero, 2025).

The idea of this approach is to associate a cooperative game with each problem. Then, a cooperative solution is computed in the associated cooperative game. Finally, the solution of the original problem is the allocation induced by the cooperative solution. Cooperative solutions can be divided in two groups. First, single-valued solutions, which propose a unique allocation for each cooperative game. Well-known examples are the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), and the τ-value (Tijs, 1981). Second, set solutions, which propose a set of allocations (which could be the empty set) for any cooperative game. Well-known examples are the core (Shapley, 1955), and the core cover (Tijs and Lipperts, 1982).

In this paper we consider the so-called tuna fishing problem introduced by Groba et al. (2020), and further studied in Bergantiños et al. (2023). The tuna industry is one of the most important

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fishing industries, both in terms of volume and revenue, and is practiced in all oceans of the world (Parker et al., 2015). Many species, such as tuna, interact with drifting objects on the surface (Dempster and Taquet, 2004). As a result, the industry began working with electronic Fish Aggregating Devices (FADs) to increase the number of floating objects in the ocean and facilitate their detection (Davies et al., 2014).

Tropical tuna vessels usually work as follows. First, each vessel (actually, the skipper) releases its FADs into the ocean, in some specific positions chosen by the skipper. Later, each vessel retrieves only its own FADs. It should be noted that an important part of the skipper's salary depends on the catches made. In addition, the firm that owns the tuna vessels is responsible for all operating costs, of which fuel consumption is by far the most important. Taking into account the working operation, any firm would be interested in the possibility of sharing and reallocating FADs among the vessels that make up its fleet. This reallocation is possible due to the satellite technology of the FADs. When FADs are shared among vessels, fuel consumption is reduced, resulting in additional revenue, and a reduction in CO_2 emissions. Since the reallocation of FADs affects the amount of tuna caught by each vessel (and thus crew wages), it is of utmost importance to find compensation mechanisms that encourage cooperation among vessels.

Bergantiños et al. (2023) associates every tuna fishing problem with a cooperative game. They prove that the cooperative game is superadditive. They consider the Shapley value. The main result is an implementation of the Shapley value. Namely, they consider a non-cooperative game and prove that the payoffs of the equilibria of the non-cooperative game are related to the Shapley value of the cooperative game.

In this paper we study the cooperative game introduced in Bergantiños et al. (2023). In our first results we prove that the core is non-empty. The τ-value is one of the most popular single solutions in cooperative game theory. It was introduced in Tijs (1981) and later studied later in many papers such as Driessen and Tijs (1983), Driessen and Tijs (1985), Tijs (1987), and Tijs and Otten (1993). We prove that the τ-value exists and we give an explicit formula for calculating it. This is an advantage over the Shapley value, which does not have an explicit formula and should then be computed as an average of the marginal contributions.

The cooperative game can be decomposed as the sum of an additive game and a cooperative game called the distance game. The distance game corresponds to the distances saved when agents cooperate and share their FADs. We prove that the distance game is a generalized big boss game (Bahel, 2016), but not a big boss game (Muto et al., 1988).

Groba et al. (2020) argue that tuna vessels usually work in groups of two or three vessels belonging to the same firm. Thus we study the cooperative game where the number of agents is three (the firm and two vessels) or four (the firm and three vessels). The nucleolus is also a well-known single value in cooperative games introduced in Schmeidler (1969) and studied later in Maschler et al. (1992) and Brânzei et al. (2006). We first prove that when there are two vessels then the τ-value and the nucleolus coincide. In the case of three vessels we study when the cooperative game is a big boss game. We also provide an explicit formula for the nucleolus.

The paper is organized as follows. In Section 2 we introduce the tuna fishing problem. In Section 3 we we study the core and the τ-value of the cooperative game associated with the tuna fishing problem. In Section 4 we study the distance game. In Section 5 we study the case of two and three vessels. Section 6 concludes.

2. The tuna fishing vessels problem

The tuna fishing vessels problem has been considered in Groba et al. (2020) and Bergantiños et al. (2023), where it is explained in detail. We explain it briefly. It is inspired by the case of tropical tuna vessels, which usually work in small groups (two or three) belonging to the same firm. They work as follows. First, each vessel (actually, the skipper) releases its FADs into the ocean, in some specific positions chosen by the skipper based on his/her experience. Later, each vessel retrieves only its own FADs. It should be noted that an important part of the skipper's salary depends on the catches made. In addition, the firm that owns the tuna vessels is responsible for all operating costs, of which fuel consumption is by far the most important. We now formally introduce the problem.

Given a finite set X we denote by Π_X be the set of all orders over X.

Let $N = \{1, \ldots, n\}$ be the set of tuna vessels, working for the same firm f. For any $S \subseteq N$, we denote $S_f = S \cup \{f\}$. There is a finite number of FADs that have been assigned to the vessels. The set of all FADs is $B = \{b_1, ..., b_m\}$, and for each $b \in B$, $\alpha(b) \in N$ denotes the vessel to which FAD b is initially assigned. Thus, each vessel $i \in N$ has an initial endowment of FADs

$$B_i^\alpha = \{b \in B : \alpha(b) = i\}.$$

Given $S \subseteq N$, $d(S)$ denotes the minimum distance that vessels in S have to travel for recovering all FADs in $\bigcup_{i \in S} B_i^\alpha$. Below we show in detail how to calculate $d(S)$.

Let $\varrho : \bigcup_{i \in S} B_i^\alpha \longrightarrow S$ be a function that reassigns the FADs initially assigned to vessels in S among themselves. $R\left(\bigcup_{i \in S} B_i^\alpha\right)$ is the set of all possible reassignment functions.

For each vessel $i \in S$, B_i^ϱ denotes the FADs assigned by ϱ to vessel i:

$$B_i^\varrho = \left\{ b \in \bigcup_{i \in S} B_i^\alpha : \varrho(b) = i \right\}.$$

We denote by $d^\varrho(i, \pi)$ the distance traveled by vessel i to recover all the FADs assigned to it by ϱ, following the order $\pi \in \Pi_{B_i^\varrho}$. Then, the minimum distance traveled by vessel i to recover all FADs in B_i^ϱ is computed by minimizing $d^\varrho(i, \pi)$ on the orders:

$$d^\varrho(i) = \min \left\{ d^\varrho(i, \pi) : \pi \in \Pi_{B_i^\varrho} \right\}.$$

Let $d^\varrho(S)$ denote the distance traveled by all vessels in S to recover the FADs in $\bigcup_{i \in S} B_i^\alpha$, when each vessel recovers the FADs assigned by ϱ:

$$d^\varrho(S) = \sum_{i \in S} d^\varrho(i).$$

Finally, $d(S)$ is computed by minimizing $d^\varrho(S)$ on the reassignment functions:

$$d(S) = \min \left\{ d^\varrho(S) : \varrho \in R\left(\bigcup_{i \in S} B_i^\alpha \right) \right\}.$$

It is obvious that for all $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$d(S \cup T) \leq d(S) + d(T). \tag{1}$$

Note that $d(i)$ represents the original distance traveled by vessel i to recover all its originally assigned FADs.

We make the following assumptions:

1. The firm knows the location of all vessels and all FADs. Each vessel knows the location of all its assigned FADs and does not know the location of the FADs assigned to other vessels.

2. Each vessel has a cost c per mile traveled, and this cost is paid for by the firm.

3. $q = \{q(b)\}_{b \in B}$ is the amount of tuna recovered in FAD b and it's only known after fishing. The total amount of tuna recovered in the FADs of vessel i is $q_i^\alpha = \sum_{b \in B_i^\alpha} q(b)$.

4. Each vessel receives a price p for each ton of tuna recovered.

5. The firm sells all tuna collected from the vessels for a price p_f for each ton of tuna.

6. Every vessel generates revenue to the firm: $pq_i^\alpha + cd(i) \leq p_f q_i^\alpha$, for each $i \in N$.

Then, the total revenue is the difference between the profit from the sale of all the tuna minus the cost of the fuel. So, without sharing the FADs, this is

$$p_f \sum_{i \in N} q_i^\alpha - c \sum_{i \in N} d(i).$$

A **tuna fishing vessels problem**, briefly a problem, is a tuple $P = (N_f, B, \alpha, c, q, p, p_f)$ where the elements of P are defined as above.

For every $i \in N_f$, let $g^P(i)$ be the revenue obtained without sharing the FADs. Formally, for each $i \in N_f$,

$$g^P(i) = \begin{cases} (p_f - p) \sum_{j \in N} q_j^\alpha - c \sum_{j \in N} d(j) & \text{if } i = f, \\ pq_i^\alpha & \text{otherwise.} \end{cases}$$

When no confusion arises we write $g(i)$ instead of $g^P(i)$.

The following example, introduced in Bergantiños et al. (2023), illustrates the concepts presented above.

Example 1. Let $P = (N_f, B, \alpha, c, q, p, p_f)$ be such that

- $N = \{1, 2\}$.
- $B = \{b_1, b_2\}$. The vessels and FADs are located in a line, from left to right. The distance between Vessel 1 and FAD b_2 is 500; the distance between FADs b_2 and b_1 is 300; and the distance between FAD b_1 and Vessel 2 is 700.
- $\alpha(b_1) = 1$ and $\alpha(b_2) = 2$.
- $c = 29$.

- $q(b_1) = 110$, and $q(b_2) = 130$.
- $p = 140$, and $p_f = 1400$.

It is straightforward to see that:

- $d(1) = 800$, $d(2) = 1000$, $\varrho(b_1) = 2$, $\varrho(b_2) = 1$, and $d(1,2) = 1200$.
- The total revenue is $1400(110 + 130) - 29(800 + 1000) = 283800$.
- The revenue without sharing the FADs is $g(1) = 140(110) = 15400$, $g(2) = 140(130) = 18200$, and $g(f) = (1400 - 140)240 - 29(1800) = 250200$.

3. The cooperative game approach

In this section we consider the cooperative approach to the tuna fishing vessels problem. We first introduce some well-kown concepts of cooperative games. Later, we introduce the cooperative game associated with a tuna fishing vessels problem following Bergantiños et al. (2023). We prove that the core of this game is non-empty and we give an expression for the τ-value.

3.1. Preliminaries

A **transferable utility game** (thereafter TU game) is a pair (N, v) where $N \subset \mathbb{N}$ is the finite set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is the **characteristic function**. Any subset S of N is a **coalition**, and $v(S)$ represents the worth that members of S can obtain if they cooperate. Henceforth, the singleton $\{i\}$ is denoted by i, and for any $S \subseteq N$, $|S| = s$. The coalition N is referred to as the **grand coalition**. When no confusion arises, we address v as a game.

Given a game v, an **allocation** is a vector $x \in \mathbb{R}^n$ such that $x(N) := \sum_{i \in N} x_i = v(N)$. An allocation x is an **imputation** if for each $i \in N$, $x_i \geq v(i)$. Let $I(v)$ denote the set of imputations of v.

A **solution** φ is a correspondence that associates with each game v a set $\varphi(v) \subset \mathbb{R}^N$.

The **core** of v is defined as

$$C(v) = \left\{ x \in \mathbb{R}^N : x(N) = v(N) \text{ and } \forall S \subset N, x(S) \geq v(S) \right\}.$$

For any game v and every $i \in N$, let

$$M_i(v) = v(N) - v(N \backslash i)$$

be player i's **marginal contribution** to the grand coalition. The vector $M(v) = (M_i(v))_{i \in N}$ is called the **utopia vector** of v and $M_i(v)$ is referred to as the utopia payoff of player i. Given $S \subseteq N$ containing player i, we call

$$r_i^v(S) = v(S) - \sum_{j \in S \backslash i} M_j(v)$$

the **remainder** for i in coalition S. This amount can be thought of as the payoff left for player i in coalition S after all other players have received their utopia payoffs. The **minimum right vector** is $m(v) = (m_i(v))_{i\in N}$ where $m_i(v) = \max_{S\subseteq N: i\in S} \{r_i^v(S)\}$. In the minimum right payoff, $m_i(v)$, player i gets the largest possible remainder.

The **core cover** Tijs and Lipperts (1982) of v consists of the set of allocations that gives each player at least their minimum right and at most their utopia payoff. Namely,

$$CC(v) = \left\{x \in \mathbb{R}^N : x(N) = v(N), m(v) \leq x \leq M(v)\right\}.$$

As its name suggests, the core cover contains the core of any game.

When the core cover is non-empty, the **τ-value** Tijs (1981) is defined as

$$\tau(v) = \lambda M(v) + (1-\lambda)m(v) \tag{2}$$

with $\lambda \in [0,1]$ such that $\sum_{i\in N} \tau_i(v) = v(N)$.

Given a game (N, v), the **excess** of $S \subseteq N$ with respect to any $x \in I(v)$ is defined as

$$e(S,x) = v(S) - x(S).$$

This amount can be seen as a measurement of the dissatisfaction that coalition S has when the imputation x is realized. For each $x \in I(v)$, let $\theta(x) \in R^{2^n}$ be the vector of all excesses $e(S,x)$ arranged in non-increasing order, *i.e*, $\theta_i(x) \geq \theta_j(x)$ if $1 \leq i < j \leq 2^n$. For any $x, y \in I(v)$, we say that x is **more acceptable** than y (and we write $x \succ y$) if there is an integer $1 \leq j \leq 2^n$ such that $\theta_i(x) = \theta_i(y)$ if $1 \leq i < j$ and $\theta_j(x) < \theta_j(y)$. As usual, $x \succeq y$ if either $x \succ y$ or $x = y$.

The nucleolus Schmeidler (1969) consists of those imputations such that there is no other more acceptable. Formally, the **nucleolus** of v is the set

$$\eta(v) = \{x \in I(v) | x \succeq y, \forall y \in I(v)\}.$$

It is known that if $I(v) \neq \emptyset$, the nucleolus is non-empty and contains a unique allocation. Furthermore if $C(v) \neq \emptyset$, the nucleolus belongs to the core.

3.2. The associated cooperative game

Bergantiños et al. (2023) associate a TU game (N_f, v^P) with very tuna fishing vessels problem, which reflects the revenues that can be obtained from cooperation, sharing the FADs in this case.

For each problem P, the **revenue game**, (N_f, v^P), is defined as follows. For each $S \subseteq N$,

$$v^P(S) = \sum_{i\in S} g(i) \quad \text{and} \quad v^P(S_f) = \sum_{i\in S_f} g(i) + c\left[\sum_{i\in S} d(i) - d(S)\right]. \tag{3}$$

When no confusion arises we write v instead of v^P.

Notice that for each $i \in N_f$, $v(i) = g(i)$.

Example 2 (*Continuation of Example 1*)**.** We now compute v in Example 1.

S	$v(S)$
$\{1\}$	$g(1) = 15400$
$\{2\}$	$g(2) = 18200$
$\{f\}$	$g(f) = 250200$
$\{1,2\}$	$g(1) + g(2) = 33600$
$\{1,f\}$	$g(1) + g(f) = 265600$
$\{2,f\}$	$g(2) + g(f) = 268400$
$\{1,2,f\}$	$g(1) + g(2) + g(f) + 29(800 + 1000 - 1200) = 301200$

We introduce some notation needed for some of our results. For each $S \subseteq N$,

$$e^d(S) := \sum_{i \in S}[d(N) - d(N \backslash i)] - d(S).$$

Let

$$\hat{e}^d := \max_{S \subseteq N} \left\{ e^d(S) \right\}.$$

3.3. The core of the revenue game

Next, we prove that the core of the revenue game is non-empty.

Proposition 1. *For each problem P,*

$$C(v^P) \neq \emptyset.$$

Proof. We prove that the allocation x, defined for each $i \in N_f$ as

$$x_i = \begin{cases} g(i) + c\left[\sum_{j \in N} d(j) - d(N)\right] & \text{if } i = f, \\ g(i) & \text{otherwise,} \end{cases}$$

belongs to $C(v^P)$.

Because of the definition of $v^P(N_f)$ we have that

$$\sum_{i \in N_f} x_i = v^P(N_f).$$

Let $S \subseteq N$. Since $v^P(S) = 0$ and for each $i \in N$, $v^P(i) = g(i) \geq 0$, we deduce that

$$\sum_{i \in S} x_i \geq v^P(S).$$

Moreover, we have that

$$\begin{aligned} \sum_{i \in S_f} x_i &= \sum_{i \in S_f} g(i) + c\left[\sum_{i \in N} d(i) - d(N)\right] \\ &= \sum_{i \in S_f} g(i) + c\left[\sum_{i \in S} d(i) - d(S)\right] - c\left[d(N) - \sum_{i \in N \backslash S} d(i) - d(S)\right] \\ &= v^P(S_f) - c\left[d(N) - \sum_{i \in N \backslash S} d(i) - d(S).\right] \end{aligned}$$

By (1), $d(N) - \sum_{i \in N \backslash S} d(i) - d(S) \leq 0$ and hence,

$$\sum_{i \in S_f} x_i \geq v^P(S_f).$$

■

We have proved that the core is non-empty by proving that the allocation giving to each vessel i its individual value (namely $v(i)$) and the remaining to the firm (namely, $v(N_f) - \sum_{i\in N} v(i)$) belongs to the core.

Depending on the problem P, the core could contain several elements. For instance, in Example 1, the allocation $(17000, 19000, 265200)$ also belongs to the core. Notice that in this allocation each vessel and the firm obtains strictly more than its individual value, which is the revenue when there is no cooperation (namely, the vessels do not share its FADs).

3.4. The τ-value of the revenue game

We now study the τ-value of the revenue game. Since the core is a subset of the core cover, Proposition 1 allows to conclude that the core cover is non-empty and that the τ-value exists. The following proposition gives the expression of the τ-value of v^P.

Proposition 2. *For each problem P, and for each $i \in N_f$,*

$$\tau_i(v^P) = \begin{cases} g^P(i) + \lambda c\left[\sum_{j\in N} d(j) - d(N)\right] + (1-\lambda)c\max\{0, \hat{e}^d\} & \text{if } i = f, \\ g^P(i) + \lambda c\left[d(i) + d(N\backslash i) - d(N)\right] & \text{otherwise,} \end{cases}$$

with $\lambda \in [0,1]$ such that $\sum_{i\in N_f} \tau_i(v^P) = v^P(N_f)$.

Proof. Since the problem P is the same throughout the proof, we use v instead of v^P, and g instead of g^P.

We first compute the utopia vector for the firm.

$$\begin{aligned} M_f(v) =& v(N_f) - v(N) = \sum_{j\in N_f} g(j) + c\left[\sum_{j\in N} d(j) - d(N)\right] - \sum_{j\in N} g(j) \\ =& g(f) + c\left[\sum_{j\in N} d(j) - d(N)\right]. \end{aligned}$$

Now, we compute the utopia vector for each vessel $i \in N$.

$$\begin{aligned} M_i(v) =& v(N_f) - v(N_f\backslash i) \\ =& \sum_{j\in N_f} g(j) + c\left[\sum_{j\in N} d(j) - d(N)\right] - \sum_{j\in N_f\backslash i} g(j) - c\left[\sum_{j\in N\backslash i} d(j) - d(N\backslash i)\right] \\ =& g(i) + c\left[d(i) + d(N\backslash i) - d(N)\right]. \end{aligned}$$

For each vessel $i \in N$, we compute the minimum right $m_i(v)$. Let $S \subseteq N$ be such that $i \in S$. Then,

$$\begin{aligned} r_i^v(S) =& v(S) - \sum_{j\in S\backslash i} M_j(v) \\ =& \sum_{j\in S} g(j) - \sum_{j\in S\backslash i} \left(g(j) + c\left[d(j) + d(N\backslash j) - d(N)\right]\right). \end{aligned}$$

By equation (1), for all $j \in S\backslash i$,

$$d(j) + d(N\backslash j) \geq d(N).$$

Therefore,

$$r_i^v(S) \leq v(i) = g(i).$$

Besides,

$$\begin{aligned}
r_i^v(S_f) =& v(S_f) - \sum_{j \in S_f \backslash i} M_j(v) \\
=& \sum_{j \in S_f} g(j) + c\left[\sum_{j \in S} d(j) - d(S)\right] - g(f) - c\left[\sum_{j \in N} d(j) - d(N)\right] \\
& - \sum_{j \in S \backslash i} \left(g(j) + c\left[d(j) + d(N\backslash j) - d(N)\right]\right) \\
=& g(i) - c\left[\sum_{j \in S \backslash i} d(N\backslash j) + \sum_{j \in N \backslash i} d(j) - |S|d(N) + d(S)\right] \\
=& g(i) - c\left[\sum_{j \in S \backslash i} \left(d(N\backslash j) + d(j) - d(N)\right) + \sum_{j \in N \backslash S} d(j) + d(S) - d(N)\right].
\end{aligned}$$

By (1) we have that $r_i^v(S_f) \leq g(i) = v(i)$. Since $r_i^v(i) = v(i)$, $m_i(v) = g(i)$.
Now, we compute the minimum right for the firm. For every $S \subseteq N$,

$$\begin{aligned}
r_f^v(S_f) =& \sum_{j \in S_f} g(j) + c\left[\sum_{j \in S} d(j) - d(S)\right] - \sum_{j \in S} \left(g(j) + c\left[d(j) + d(N\backslash j) - d(N)\right]\right) \\
=& g(f) - c\left[\sum_{j \in S} d(N\backslash j) - |S|d(N) + d(S)\right] \\
=& g(f) + ce^d(S).
\end{aligned}$$

We consider several cases:

- If $\hat{e}^d \leq 0$, then for all $S \subseteq N$, $e^d(S) \leq 0$ and

$$r_f^v(S_f) = g(f) + ce^d(S) \leq g(f).$$

  Since $r_f^v(f) = v(f) = g(f)$, $m_f(v) = g(f)$.

- If $\hat{e}^d > 0$, then there exists at least one $S \subseteq N$ such that $e^d(S) > 0$. Let $S^* = \operatorname*{argmax}_{S \subseteq N} e^d(S)$. Thus, for all $S \subseteq N$,

$$r_f^v(S_f) = g(f) + ce^d(S) \leq g(f) + ce^d(S^*) = g(f) + c\hat{e}^d.$$

  Then, $r_f^v(S_f) \leq g(f) + c\hat{e}^d$. Since $r_f^v(S_f^*) = g(f) + c\hat{e}^d$, $m_f(v) = g(f) + c\hat{e}^d$.

Hence, $m_f(v) = g(f) + c\max\{0, \hat{e}^d\}$.
Now, the formula for $\tau(v)$ is a straightforward consequence of (2).

■

By Proposition 2, the τ value can be easily computed from v. This is an advantage over the Shapley value, which has no explicit formula, making its computation NP-hard.

Example 3 (*Continuation of Example 1*)**.** We now compute the τ-value in Example 1. We already know the values of g and d. We now compute $\hat{e}^d$.

S	$e^d(S)$
$\{1\}$	$d(1,2) - d(2) - d(1) = -600$
$\{2\}$	$d(1,2) - d(1) - d(2) = -600$
$\{1,2\}$	$d(1,2) - d(2) + d(1,2) - d(1) - d(1,2) = -600$

Then, $\hat{e}^d = 0$. It is straightforward to see that for each $i \in N_f$, $\tau_i(v) = g(i) + 5800$. Thus, the revenue obtained from sharing the FADs is divided equally among all agents. This is what happens in this example, but not in general.

4. The distance game

In this section we introduce the so-called distance game, associated with the distances "saved" when FADs are shared. We study some properties of this game.

For each problem P, let us define the **distance game**, (N_f, w^P), as follows. For all $S \subseteq N$,

$$w^P(S) = 0 \quad \text{and} \quad w^P(S_f) = \sum_{i \in S} d(i) - d(S). \tag{4}$$

The worth of a coalition is the distance that can be saved if the vessels share their FADs. Note that this game is computed using distances only. When no confusion arises, we write w instead of w^P.

We have the following trivial relation with the game v. For each $S \subseteq N_f$,

$$v(S) = \sum_{i \in S} g(i) + cw(S).$$

Taking into account this relationship between the two TU games and a property that the τ-value and the nucleolus satisfy[2], we have the following remark.

Remark 1. For each problem P and for each $i \in N_f$,

- $\tau_i(v^P) = g^P(i) + c\tau_i(w^P)$ and
- $\eta_i(v^P) = g^P(i) + c\eta_i(w^P)$.

Big boss games were introduced in Muto et al. (1988). A game v is a **big boss game** with a powerful player $i^* \in N$ if it satisfies the following three conditions:

[2]The property we are referring to is usually found in the literature as *covariance*: given two games v and w, $r \in \mathbb{R}^{++}$ and $\alpha \in \mathbb{R}^N$ such that $v(S) = rw(S) + \sum_{i \in S} \alpha_i$, for all $S \subseteq N$, it is verified that $\varphi(v) = r\varphi(w) + \alpha$.

- $(B1)$ v is monotone (namely, $v(S) \leq v(T)$ when $S \subseteq T$);
- $(B2)$ $v(S) = 0$ if $i^* \notin S$; and
- $(B3)$ $v(N) - v(S) \geq \sum\limits_{i \in N \setminus S} [v(N) - v(N \setminus i)]$ if $i^* \in S$.

Bahel (2016) extends the family of big boss games considering all games that satisfy $(B1)$ and $(B2)$ but not $(B3)$ and calls this family **generalized big boss games**.

In the next proposition we prove that the distance game is a generalized big boss game but not a big boss game.

Proposition 3. *For each problem P, w^P is a generalized big boss game but not a big boss game.*

Proof. Let P be a problem and w the associated distance game. We prove that w satisfies $(B1)$ and $(B2)$ but fails $(B3)$.

It is clear that w satisfies $(B2)$ (just take $i^* = f$).

We prove that w satisfies $(B1)$. Let $S \subseteq T$. If $f \notin S$ then

$$w(T) \geq 0 = w(S).$$

If $f \in S$ then

$$w(T) = \sum_{i \in T \setminus f} d(i) - d(T \setminus f) = \sum_{i \in S \setminus f} d(i) - d(S \setminus f) + \sum_{i \in T \setminus S} d(i) + d(S \setminus f) - d(T \setminus f).$$

By (1), $\sum\limits_{i \in T \setminus S} d(i) + d(S \setminus f) - d(T \setminus f) \geq 0$. Then,

$$w(T) \geq \sum_{i \in S \setminus f} d(i) - d(S \setminus f) = w(S).$$

We prove that w does not satisfy $(B3)$. Consider a problem with three vessels. Vessel 1 has 3 FADs, 2 located in the fishing area A1 and 1 located in the fishing area A2. Vessel 2 has 3 FADs, 2 located in A1 and 1 located in A2. Vessel 3 has 2 FADs located in the fishing area A3. Vessels 1 and 2 are located initially in A1 whereas 3 is located initially in A3. The distances inside each fishing area are around 10, the distance between A1 and A2 is around 1000, the distance between A2 and A3 is around 100. This situation can be summarized in the following table:

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$d(S)$	1020	1020	20	1050	140	140	170

Take $S = \{3, f\}$. Then,

$$\begin{aligned} w(1,2,3,f) =& 1020 + 1020 + 20 - 170 = 1890 \\ w(3,f) =& 20 - 20 = 0 \\ w(1,3,f) =& 1020 + 20 - 140 = 900 \\ w(2,3,f) =& 1020 + 20 - 140 = 900 \end{aligned}$$

Then, $w(N_f) - w(S) = 1890$ and $\sum\limits_{i \in N_f \setminus S} [w(N_f) - w(N_f \setminus \{i\})] = 2(1890 - 900) = 1980$. Hence, w does not satisfy $(B3)$.

■

Regarding the core of w, we can easily see that it is non-empty: the allocation that gives $w(N_f)$ to the firm and zero to the vessels belongs to the core. The proof uses exactly the same arguments as the proof of Proposition 1 and it is omitted.

Proposition 4. *For each problem P,*

$$C(w^P) \neq \emptyset.$$

5. The case of two and three vessels

Our theoretical model is inspired in the case of tropical tuna vessels, which uses FADs as the main way of fishing. Groba et al. (2020) argue that tropical tuna vessels usually work in groups of two or three vessels belonging to the same firm. In this section we obtain additional results for these cases, which are the most relevant for the practical cases.

We begin with the almost trivial case of two vessels. Remark 1 together with the fact that in the two-vessel case all players are symmetric in w, allow us to obtain the following result for v.

Proposition 5. *For each problem P where $N = \{1, 2\}$,*

$$\tau_i(v^P) = \eta_i(v^P) = g^P(i) + \frac{c}{3}(d(1) + d(2) - d(N)), \quad \text{for all } i \in N_f.$$

Thus, in the case of two vessels, the τ-value and the nucleolus coincide. In this allocation, each agent (vessel or the firm) receives its individual revenue, i.e., $g(i)$. Additionally, the revenue obtained from sharing the FADs is divided equally.[3]

We now consider the case of three vessels. Our first result says that when the minimal right of f is strictly positive in the distance game w, w is a big boss game.

Proposition 6. *For each problem P where $N = \{1, 2, 3\}$ and $m_f(w^P) > 0$, w^P is a big boss game.*

Proof. We have already shown that conditions $(B1)$ and $(B2)$ are satisfied. It only remains to prove that $(B3)$ is satisfied. As the problem P is the same in the whole proof, we use w instead of w^P.

Since $m_f(w) = \max\limits_{S \subseteq N_f : f \in S} \left\{ r_f^w(S) \right\}$ we can rewrite it as

$$m_f(w) = \max_{S \subseteq N} \left\{ r_f^w(S_f) \right\}$$

where for each $S \subseteq N$,

[3] Bergantiños et al. (2023) prove that this allocation also coincides with the Shapley value of v^P for the two-vessel case.

$$
\begin{aligned}
r_f^w(S_f) =& w(S_f) - \sum_{i\in S} M_i(w) \\
=& \sum_{i\in S} d(i) - d(S) - \sum_{i\in S} (w(N_f) - w(N_f \setminus i)) \\
=& \sum_{i\in S} d(i) - d(S) - \sum_{i\in S} \left(\sum_{j\in N} d(j) - d(N) - \left(\sum_{j\in N\setminus i} d(j) - d(N\setminus i) \right) \right) \\
=& \sum_{i\in S} d(i) - d(S) - \sum_{i\in S} (d(i) - d(N) + d(N\setminus i)) \\
=& - d(S) + \sum_{i\in S} (d(N) - d(N\setminus i)) \\
=& e^d(S).
\end{aligned}
$$

Since $m_f(w^d) > 0$, there exists at least one $S \subseteq N$ such that $e^d(S) > 0$. Let $S^* = \underset{S\subseteq N}{\operatorname{argmax}}\, e^d(S)$. Assume that $S^* = \{i\}$, for some $i \in N$. Then

$$e^d(S^*) > 0 \iff d(N) - d(N\backslash i) - d(i) > 0 \iff d(N) > d(i) + d(N\backslash i),$$

a contradiction by (1).

Now, assume that $|S^*| = 2$ and let $N\backslash S^* = \{j\}$. Then we have that

$$e^d(S^*) > 0 \iff 2d(N) - \sum_{i\in S^*} d(N\backslash i) - d(N\backslash j) > 0 \iff 2d(N) - \sum_{i\in N} d(N\backslash i) > 0.$$

We now prove that

$$2d(N) - \sum_{i\in N} d(N\backslash i) > 0. \tag{5}$$

If $S^* = N$, we get the same inequality as in (5):

$$e^d(N) > 0 \iff 3d(N) - \sum_{i\in N} d(N\backslash i) - d(N) > 0 \iff 2d(N) - \sum_{i\in N} d(N\backslash i) > 0.$$

Now, let $S \subseteq N$. On the one hand, we have that

$$w(N_f) - w(S_f) = \sum_{i\in N} d(i) - d(N) - \left[\sum_{i\in S} d(i) - d(S) \right] = \sum_{i\in N\backslash S} d(i) - d(N) + d(S). \tag{6}$$

On the other hand,

$$
\begin{aligned}
\sum_{i\in N_f\backslash S_f} [w(N_f) - w(N_f\backslash i)] &= \sum_{i\in N\backslash S} \left[\sum_{j\in N} d(j) - d(N) - \sum_{j\in N\backslash i} d(j) + d(N\backslash i) \right] \\
&= \sum_{i\in N\backslash S} [d(i) - d(N) + d(N\backslash i)]\,. \qquad (7)
\end{aligned}
$$

We now prove that $(B3)$ holds. We consider several cases depending on the cardinality of S.

- If $|S| = 0$, we have that

$$\begin{aligned}
w(N_f) - w(S_f) &\overset{(6)}{=} \sum_{i \in N} d(i) - d(N) \\
&\overset{(5)}{>} \sum_{i \in N} d(i) - d(N) + \sum_{i \in N} d(N \backslash i) - 2d(N) \\
&= \sum_{i \in N} [d(i) - d(N) + d(N \backslash i)] \\
&\overset{(7)}{=} \sum_{i \in N_f \backslash S_f} [w(N_f) - w(N_f \backslash i)].
\end{aligned}$$

- If $|S| = 1$, let $S = \{k\}$. Then,

$$w(N_f) - w(S_f) \overset{(6)}{=} \sum_{i \in N \backslash k} d(i) - d(N) + d(k) = \sum_{i \in N} d(i) - d(N),$$

and

$$\sum_{i \in N_f \backslash S_f} [w(N_f) - w(N_f \backslash i)] \overset{(7)}{=} \sum_{i \in N \backslash k} d(i) - 2d(N) + \sum_{i \in N \backslash k} d(N \backslash i).$$

Considering the right sides of the last two equations, we get that

$$w(N_f) - w(S_f) > \sum_{i \in N_f \backslash S_f} [w(N_f) - w(N_f \backslash i)] \iff d(k) + d(N) > \sum_{i \in N \backslash k} d(N \backslash i). \qquad (8)$$

By (1)

$$d(k) \geq d(N) - d(N \backslash k),$$

then,

$$d(k) + d(N) \geq 2d(N) - d(N \backslash k).$$

By (5)

$$2d(N) - d(N \backslash k) > \sum_{i \in N} d(N \backslash i) - d(N \backslash k) = \sum_{i \in N \backslash k} d(N \backslash i).$$

- If $|S| = 2$, let $S = N \backslash k$. Then,

$$\begin{aligned}
w(N_f) - w(S_f) &\overset{(6)}{=} d(k) - d(N) + d(N \backslash k) \\
&= \sum_{i \in N \backslash S} d(i) - d(N) + d(N \backslash k) \\
&= \sum_{i \in N \backslash S} [d(i) - d(N) + d(N \backslash i)] \\
&\overset{(7)}{=} \sum_{i \in N_f \backslash S_f} [w(N_f) - w(N_f \backslash i)].
\end{aligned}$$

- If $|S|=3$, it is easy to see that (6) and (7) are equal to zero.

Therefore, w satisfies condition $(B3)$, and it is a big boss game. ■

In our next result we prove that the core and the core cover of w coincide.

Proposition 7. *For each problem P where $N=\{1,2,3\}$,*

$$C(w^P)=CC(w^P).$$

Proof. Since the core is a subset of the core cover, only the inclusion $CC(w^P)\subseteq C(w^P)$ needs to be proved. Again, we write w instead of w^P. Take $x\in CC(w)$. We prove that $x\in C(w)$.

Let $S\subseteq N$. For each $i\in N$, $x_i\geq m_i(w)\geq w(i)=0$. Then

$$x(S)\geq 0=w(S).$$

Assume that $s\leq 1$. Then, $w(S_f)=0$. Hence

$$x(S_f)\geq x(f)\geq m_f(w)\geq w(f)=0.$$

Assume now that $s\geq 2$. Since $x(N\backslash S)\geq 0$ and (1)

$$\begin{aligned}
x(N_f)=&w(S)-x(N\backslash S)=\sum_{i\in N}d(i)-d(N)-x(N\backslash S)\\
\geq&\sum_{i\in N}d(i)-d(N)-\sum_{i\in N\backslash S}\left[d(i)+d(N\backslash i)-d(N)\right]\\
=&\sum_{i\in S}d(i)-d(N)+\sum_{i\in N\backslash S}\left[d(N)-d(N\backslash i)\right].
\end{aligned}\tag{9}$$

We now prove that equation (9) is greater or equal than $w(S_f)$. We consider two cases.

- $|S|=2$. Then $|N\backslash S|=1$. Let $N\backslash S=\{k\}$. Thus,

$$x(S_f)\geq\sum_{i\in S}d(i)-d(N)+d(N)-d(N\backslash k)=\sum_{i\in S}d(i)-d(S)=w(S_f).$$

- $|S|=3$. Then,

$$x(S_f)=x(N_f)\geq\sum_{i\in N}d(i)-d(N)=w(N_f)=w(S_f).$$

Therefore, $x\in C(w)$.

■

In big boss games, the τ-value coincides with the nucleolus, and both are the center of the core (see Muto et al., 1988). This allocation assigns to each non-big boss player half of their marginal contribution to the grand coalition, and the rest to the big boss. Below we give an expression for $\tau(w)$ and $\eta(w)$ as a consequence of Proposition 6.

Corollary 1. *Let P be a problem where $N = \{1,2,3\}$ and $m_f(w^P) > 0$. Then, for each $i \in N_f$,*

$$\tau_i(w^P) = \eta_i(w^P) = \begin{cases} \frac{1}{2}d(N) + \frac{1}{2}\sum_{j\in N}[d(j) - d(N\backslash j)] & \text{if } i = f, \\ \frac{d(i) + d(N\backslash i) - d(N)}{2} & \text{otherwise.} \end{cases}$$

We end this section by computing the nucleolus in the general case of three vessels.

Now, we can give an expression of $\eta(v)$ for the three-vessel case.

Proposition 8. *Let P be such that $N = \{1,2,3\}$. For each $i \in N_f$,*

$$\eta_i(v^P) = \begin{cases} g^P(i) + c\mu & \text{if } i = f, \\ g^P(i) + c\min\left\{\mu, \frac{d(i) + d(N\backslash i) - d(N)}{2}\right\} & \text{otherwise,} \end{cases}$$

with $\mu \geq 0$ such that $\sum_{i\in N_f} \eta_i(v^P) = v^P(N_f)$.

Proof. We first introduce a lemma that will be used in the proof. We introduce bankruptcy problems. Let N be a finite set of agents. Each agent $i \in N$ has a claim $c_i \in \mathbb{R}_+$ over an estate $E \in \mathbb{R}^+$. A bankruptcy problem on N is a pair (E, c), where E is the estate and $c = (c_i)_{i\in N}$ is a vector of claims, with $0 \leq E \leq \sum_{i\in N} c_i$.

A **bankruptcy rule** is a function f that assigns to each bankruptcy problem (E, c) a vector $f(E, c) \in \mathbb{R}^n$ such that $\sum_{i\in N} f_i(E, c) = E$ and $0 \leq f(E, c) \leq c$.

For every bankruptcy problem, the **Talmud rule**, $TAL(E, c)$ is defined, for every $i \in N$, as

$$TAL_i(E, c) = \begin{cases} \min\left\{\lambda, \frac{1}{2}c_i\right\} & \text{if } E \leq \frac{1}{2}\sum_{j\in N} c_j, \\ \max\left\{c_i - \mu, \frac{1}{2}c_i\right\} & \text{otherwise,} \end{cases}$$

where λ and μ are chosen such that $\sum_{i\in N} TAL_i(E, c) = E$.

Lemma 1. *Quant et al. (2005) Let (N, v) be a game such that $CC(v) = C(v) \neq \emptyset$. Then,*

$$\eta(v) = m(v) + TAL\left(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)\right).$$

We know by Remark 1 that $\eta_i(v) = g(i) + c\eta_i(w)$, for each $i \in N_f$. Then, we will analyze $\eta(w)$, taking into account the two possible cases for the value of $m_f(w)$.

- $m_f(w) > 0$. By inequality (5) and equation (1), we have that, for each $i \in N$,

$$\sum_{j\in N\backslash i} d(j) - d(N\backslash i) \geq \sum_{j\in N} d(N\backslash j) - 2d(N)$$

$$\implies d(N) + \sum_{j\in N} d(j) - \sum_{j\in N} d(N\backslash j) \geq d(i) + d(N\backslash i) - d(N)$$
$$\implies \frac{1}{2}d(N) + \frac{1}{2}\sum_{j\in N}[d(j) - d(N\backslash j)] \geq \frac{d(i) + d(N\backslash i) - d(N)}{2}.$$

Therefore, by Corollary 1 taking

$$\mu = \frac{1}{2}d(N) + \frac{1}{2}\sum_{j\in N}[d(j) - d(N\backslash j)]$$

we have the desired expression.

- $m_f(w) = 0$. We prove that $m_i(w) = 0$ for all $i \in N$. We first compute $M(w)$. Given $i \in N$,

$$\begin{aligned} M_i(w) =& w(N_f) - w(N_f\backslash i) \\ =& \sum_{j\in N} d(j) - d(N) - \left[\sum_{j\in N\backslash i} d(j) - d(N\backslash i)\right] \\ =& d(i) + d(N\backslash i) - d(N). \end{aligned}$$

Besides,

$$M_f(w) = w(N_f) - w(N) = w(N_f) = \sum_{j\in N} d(j) - d(N).$$

Let $S \subseteq N$ and $i \in S$. Since $w(S) = 0$, we deduce that $r_i^w(S) \leq 0$. If $S = \{i\}$, then $r_i^w(S) = 0$. Besides,

$$\begin{aligned} r_i^w(S_f) =& w(S_f) - \sum_{j\in S_f\backslash i} M_j(v) \\ =& \sum_{j\in S} d(j) - d(S) - \sum_{j\in S\backslash i} ([d(j) + d(N\backslash j) - d(N)]) - \left(\sum_{j\in N} d(j) - d(N)\right) \\ =& |S|d(N) - d(S) - \sum_{j\in S\backslash i} d(N\backslash j) - \sum_{j\in N\backslash i} d(j) \\ =& \sum_{j\in S\backslash i} [d(N) - d(N\backslash j) - d(j)] + \left[d(N) - d(S) - \sum_{j\in N\backslash S} d(j)\right]. \end{aligned}$$

By (1), $r_i^w(S_f) \leq 0$ and hence $m_i(w) = 0$.

By Proposition 7 and Lemma 1, for each $i \in N_f$,

$$\begin{aligned} \eta_i(w) =& m_i(w) + TAL_i\left(w(N_f) - \sum_{i\in N_f} m_i(w), M(w) - m(w)\right) \\ =& TAL_i\left(w(N_f), M(w)\right). \end{aligned}$$

We now prove that $w(N_f) \leq \frac{1}{2} \sum_{i \in N_f} M_i(w)$.

$$\begin{aligned} w(N_f) \leq \frac{1}{2} \sum_{i \in N_f} M_i(w) &\iff w(N_f) \leq \sum_{i \in N} M_i(w) \\ &\iff \sum_{i \in N} d(i) - d(N) \leq \sum_{i \in N} [d(i) + d(N \backslash i) - d(N)] \\ &\iff 2d(N) \leq \sum_{i \in N} d(N \backslash i). \end{aligned}$$

Since $m_f(w^d) = 0$, $e^d(S) \leq 0$ for all $S \subseteq N$. In particular,

$$e^d(N) = \sum_{i \in N} [d(N) - d(N \backslash i)] \leq 0 \implies \sum_{i \in N} d(N \backslash i) \geq 3d(N) \geq 2d(N).$$

By Lemma 1,

$$\eta_i(w) = \begin{cases} \min\left\{\mu, \frac{1}{2}\left[\sum_{j \in N} d(j) - d(N)\right]\right\} & \text{if } i = f, \\ \min\left\{\mu, \frac{d(i) + d(N \backslash i) - d(N)}{2}\right\} & \text{otherwise.} \end{cases}$$

where μ is such that $\sum_{i \in N_f} \eta_i(w) = w(N_f) = \sum_{j \in N} d(j) - d(N)$.

By (1), for each $i \in N$,

$$\begin{aligned} &\sum_{j \in N \backslash i} d(j) \geq d(N \backslash i), \\ \implies &\sum_{j \in N} d(j) - d(i) \geq d(N \backslash i), \\ \implies &\sum_{j \in N} d(j) - d(N) \geq d(i) + d(N \backslash i) - d(N). \end{aligned}$$

Asume that $\mu > \frac{1}{2}\left[\sum_{j \in N} d(j) - d(N)\right]$. Then, $\mu > \frac{d(i) + d(N \backslash i) - d(N)}{2}$ for all $i \in N$. Hence,

$$\sum_{i \in N_f} \eta_i(w) = \sum_{i \in N_f} M_i(w) \geq 2w(N_f) > w(N_f),$$

which is a contradiction.

Then, $\eta_f(w) = \mu$.

■

6. Conclusions

We have studied the cooperative game associated with the tuna fishing problem. We have proved that the core is non-empty. Furthermore, we have provided an explicit formula for the computation of the τ-value which allows us to compute it easily from the cooperative game. We have also analyzed the associated distance game and proved that it is a generalized big boss game. Finally we have considered the case of two or three vessels, the most common cases in practice. We have computed the nucleolus and we have proved that the core and the core cover coincide.

References

Bahel, E., 2016. On the core and bargaining set of a veto game. International Journal of Game Theory 45, 543–566. https://doi.org/10.1007/s00182-015-0469-7.

Bergantiños, G., Groba, C., Sartal, A., 2023. Applying the shapley value to the tuna fishery. European Journal of Operational Research 309, 306–318.

Bergantiños, G., Moreno-Ternero, J.D., 2023. Axiomatic characterizations of the core and the shapley value of the broadcasting game. International Journal of Game Theory 53, 977–988.

Bergantiños, G., Moreno-Ternero, J.D., 2025. Revenue sharing at music streaming platformss. Management Science forthcoming.

Bergantiños, G., Navarro-Ramos, A., 2023. Cooperative approach to a location problem with agglomeration economies. International Journal of Game Theory 52, 63–92.

Brânzei, R., Iñarra, E., Tijs, S., Zarzuelo, J., 2006. A simple algorithm for the nucleolus of airport profit games. International Journal of Game Theory 34, 259–272. Springer.

Curiel, I., Maschler, M., Tijs, S., 1987. Bankruptcy games. Mathematical Methods of Operations Research 31, 143–159.

Curiel, I., Pederzoli, G., Tijs, S., 1989. Sequencing games. European Journal of Operational Research 40, 344–351.

Davies, T.K., Mees, C.C., Milner-Gulland, E., 2014. The past, present and future use of drifting fish aggregating devices (fads) in the indian ocean. Marine Policy 45, 163–170.

Dempster, T., Taquet, M., 2004. Fish aggregation device (fad) research: gaps in current knowledge and future directions for ecological studies. Reviews in Fish Biology and Fisheries 14, 21–42.

Driessen, T., Tijs, S., 1983. The t-value, the nucleolus and the core for a subclass of games. Methods of operations research 46, 395–406.

Driessen, T., Tijs, S., 1985. The τ-value, the core and semiconvex games. International Journal of Game Theory 14, 229–247.

Gonçalves-Dosantos, J., Martínez, R., Sánchez-Soriano, J., 2025. Revenue distribution in streaming. Omega 132, 103233.

Groba, C., Sartal, A., Bergantiños, G., 2020. Optimization of tuna fishing logistic routes through information sharing policies: A game theory-based approach. Marine Policy 113, 103795.

López, J., Moreno, G., Sancristobal, I., Murua, J.., 2014. Evolution and current state of the technology of echo-sounder buoys used by spanish tropical tuna purse seiners in the atlantic, indian and pacific oceans. Fisheries Research 155, 127–137.

Maschler, M., Potters, J., Tijs, S., 1992. The general nucleolus and the reduced game property. International Journal of Game Theory 21, 85–106. Springer.

Muto, S., Nakayama, M., Potters, J., Tijs, S.H., 1988. On big boss games. The economic studies quarterly 39, 303–321. https://doi.org/10.11398/economics1986.39.303.

Navarro-Ramos, A., 2022. A new approach to agglomeration problems. Operations Research Letters 50, 639–645.

Norde, H., Moretti, S., Tijs, S., 2004. Minimum cost spanning tree games and population monotonic allocation schemes. European Journal of Operational Research 154, 84–97.

van den Nouweland, A., Borm, P., van Golstein Brouwers, W., Groot Bruinderink, R., Tijs, S., 1996. A game theoretic approach to problems in telecommunications. Management Science 42, 294–303.

Parker, R.W., Vázquez-Rowe, I., Tyedmers, P.H., 2015. Fuel performance and carbon footprint of the global purse seine tuna fleet. Journal of Cleaner Productiong 103, 517–524.

Potters, J., Curiel, I., Tijs, S., 1992. Traveling salesman games. Mathematical Programming 53, 199–211.

Quant, M., Borm, P., Reijnierse, H., Velzen, B.v., 2005. The core cover in relation to the nucleolus and the weber set. International Journal of Game Theory 33, 491–503.

Schlicher, L., Dietzenbacher, B., Musegaas, M., 2024. Stable streaming platforms: a cooperative game approach. Omega 125, 103020.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on applied mathematics 17, 1163–1170. https://doi.org/10.1137/0117107.

Shapley, L.S., 1953. A value for n-person games. Contributions to the Theory of Games 2, 307–317.

Shapley, L.S., 1955. Markets as cooperative games. RAND Corporaton Paper P-629, 1–5.

Tijs, S., 1981. Bounds for the core of a game and the t-value, in: Moeschlin, O., Pallaschke, D. (Eds.), Game Theory and Mathematical Economics. North-Holland Publishing Company, pp. 123–132.

Tijs, S., 1987. An axiomatization of the τ-value. Mathematical Social Sciences 13, 177–181.

Tijs, S., Driessen, T., 1986. Game theory and cost allocation problems. Management Science 32, 1015–1028.

Tijs, S., Lipperts, F., 1982. The hypercube and the core cover of n-person cooperative games. Cahiers du Centre d'Études de Researche Opérationelle 24, 27–37.

Tijs, S., Otten, G., 1993. Compromise values in cooperative game theory. TOP 1, 1–36.