Minimum cost spanning tree problems with multiple sources: the folk rule*

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Abstract

In this paper we introduce minimum cost spanning tree problems with multiple sources. We extend various definitions of the folk rule (defined for classical minimum cost spanning tree problems) to this new context.

Keywords: minimum cost spanning tree problems, multiple sources, folk rule.

1 Introduction

A group of agents is interested in goods or services provided by several suppliers, also called sources. Agents will be served through costly connections. They do not care whether they are connected directly or indirectly to the sources, but they want to be connected to all of them. This may occur for safety reasons. Consider that all suppliers offer exactly the same resource: agents will have greater assurances of service in the sense that can still enjoy the resource even if one or more sources cease to provide it. There could also be a situation where the suppliers offer different services (Internet, cable TV, etc.) and agents are interested in all of them. These situations generalize classical minimum cost spanning tree problems where there is a single source.

Given a cost spanning tree problem with multiple sources, the least costly connection (a minimal tree) that provides all the agents with the resource, taking into

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account that the agents want to be connected to every source, must be sought. This tree can be obtained, in a polynomial time, by using the same algorithms as in the classical minimum cost spanning tree problem, for instance that of Prim (1956) or that of Kruskal (1957). Nevertheless, some variants of this problem are not so easy from a computational point of view, e.g. the fixed cost spanning forest problem where the service is provided by sources yet to be constructed at a fixed cost studied in Granot and Granot (1992); the multi-source spanning tree problem studied in Farley *et al.* (2000), where the objective is to compute the spanning tree that minimizes the sum of the distances from each source to every other node; and the hop constrained Steiner trees with multiple root nodes studied in Gouveia *et al.* (2014).

Once it is known how to connect all agents to all sources at the minimum cost, another important issue that usually arises is how to allocate that cost to the various agents. Our paper studies this issue in minimum cost spanning tree problems with multiple sources. Even though in the classical setting (a single source) there are many papers in the literature on Operations Research or Economics that study how to allocate the total cost, there are few devoted to this issue in the setting of multiple sources. Two of them are mentioned below.

Rosenthal (1987) introduces the minimum cost spanning forest game where there are several sources that offer the same service and agents want to be connected to at least one source. He associates a cooperative game with this problem, studies the core and proves that it is non-empty. Kuipers (1997) studies a problem where there are multiple sources, each of them offering a different service, and each agent specifies the set of sources that she wants to be connected to. He associates a cooperative game with this problem and discusses the conditions for the non-emptiness of the core.

Our approach is different because we want all agents to be connected to all sources. From this perspective our case can be seen as a particular case of the model by Kuipers (1997) where all agents demand to be connected to every source. Nevertheless, the cooperative game that we set up to study this problem is different. In both the papers mentioned above the cost of a coalition S is the minimum cost of connecting all members in S to every source. We follow the standard approach (as in classical minimum cost spanning tree problems) and assume that agents in S can not use the locations of agents outside S. Kuipers (1997) follows the monotone extension proposed by Bird (1976) and assumes that agents in S can use the location of agents outside the coalition.

In classical minimum cost spanning tree problems the most popular rule is the so called "folk rule". The folk rule has been proved to satisfy very appealing properties. It provides allocations in the core and is monotonic in the population and in the cost matrix. It is also additive in the cost matrix, which makes it easy to compute. Our aim is to extend the definition of the folk rule to our setting. There are many different but equivalent ways of defining the folk rule in the classical model, some of which are outlined below:

1. As the Shapley value of the irreducible game (See Bergantiños and Vidal-Puga (2007a));

2. As an obligation rule (Tijs *et al* (2006) and Bergantiños and Kar (2010));

3. As a partition rule (Bergantiños *et al.* (2010, 2011));

4. Through a cone-wise decomposition (Branzei *et al.* (2004) and Bergantiños and Vidal-Puga (2009));

5. As the Shapley value of the optimistic game (Bergantiños and Vidal-Puga (2007b) and Bergantiños and Lorenzo (2008));

6. Through Boruvka's algorithm (Bergantiños and Vidal-Puga (2011)); and

7. Through a painting procedure (Bergantiños *et al.* (2014)).

In this paper we extend definitions one, two, three, and four to our setting. Our main result is that all four rules provide the same allocation.

The paper is structured as follows. In Section 2 we introduce minimum cost spanning tree problems with multiple sources and explain our notation. In Section 3 we extend the four definitions of the folk rule to our setting. Finally, in Section 4 we prove that the four definitions coincide in our setting.

2 The model

We are interested in networks whose nodes are elements of a set $N \cup M$, where N is the set of agents and M is the set of sources. Usually we take $N = \{1, ..., |N|\}$ and $M = \{s_1, ..., s_{|M|}\}$ where $|N|$ and $|M|$ denote the cardinality of N and M respectively.

A *cost matrix* $C = (c_{ij})_{i,j \in N \cup M}$ on $N \cup M$ represents the cost of a direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N \cup M$ and $c_{ii} = 0$ for each $i \in N \cup M$. Since $c_{ij} = c_{ji}$, we will work with undirected arcs $\{i, j\}$. We denote the set of all cost matrices over $N \cup M$ as $\mathcal{C}^{N \cup M}$. Given C, $C' \in \mathcal{C}^{N \cup M}$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N \cup M$. Analogously, given $x, y \in \mathbb{R}^N$, we say $x \leq y$ if $x_i \leq y_i$ for all $i \in N$.

A *minimum cost spanning tree problem with multiple sources* (a "problem" for short) is characterized by a triple (N, M, C) where N is the set of agents, M is the set of sources, and C is the cost matrix. Given a subset $S \subset N$ we denote by (S, M, C) the restriction of the problem to the subset of agents S.

Note that classical minimum cost spanning tree problems, referred to as *mcstp* for short, correspond to the case where M has a single element, which is denoted by 0. A *mcstp* is characterized by a pair (N_0, C) where $N_0 = N \cup \{0\}$ and C denotes the cost matrix.

For each network g and each pair of distinct nodes i and j, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{\{i_{s-1}, i_s\}\}_{s=1}^{p}$ that satisfy $\{i_{s-1}, i_s\} \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$, and $j = i_p$. A *cycle* is a path from i to i. For each $i, j \in N \cup M$, *i and j are connected in g* if there is a path from i to j. A *tree* is a connected network with no cycles.

For each network g, $S \subset N \cup M$ is a *connected component* if two conditions hold: firstly, for each $i, j \in S$, i and j are connected in g; secondly, S is maximal, *i.e.*, for each $T \subset N \cup M$ with $S \subsetneq T$, there are $i, j \in T$, $i \neq j$, such that i and j are not connected in g. Let $P(g) = \{S_k(g)\}_{k=1}^{n(g)}$ be the partition of $N \cup M$ in *connected components* induced by g. Given a network g, let $S(P(g), i)$ denote the element of $P(g)$ to which i belongs.

Let $P(N \cup M)$ denote the set of all partitions of $N \cup M$. Let $P = \{S_1, \ldots, S_m\}$ be a generic element of $P(N \cup M)$. Given $P, P' \in P(N \cup M)$, P is *finer* than P' if for each $S \in P$ there is $T \in P'$ such that $S \subset T$.

Given a finite set S we denote by $\Delta(S)$ the simplex over S.

Given a problem (N, M, C) and a network g, the cost associated with g is defined as $c(N, M, C, g) = \sum_{\{i,j\} \in g} c_{ij}$. When there are no ambiguities, we write $c(g)$ or $c(C, g)$ instead of $c(N, M, C, g)$.

Our initial objective is to minimize the cost of connecting all agents to the sources. This is achieved by a *minimal tree*. Formally, a tree t is a minimal tree if $c(t) = \min\{c(g) : g \text{ is a tree}\}$. There is always a minimal tree but it does not necessarily have to be unique. The algorithm of Kruskal (1956) enables it to be computed. The idea behind Kruskal's algorithm is to construct a minimal tree by sequentially adding the cheapest arc avoiding cycles.

Formally, let $A(C) = \{\{i, j\} : i, j \in N \cup M \text{ and } i \neq j\}$ and $g^0(C) = \emptyset$.

Stage 1. Take an arc $\{i, j\} \in A(C)$ such that $c_{ij} = \min_{\{k,l\} \in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Now $\{i^1(C), j^1(C)\} = \{i, j\}$, $A(C) = A(C) \setminus \{i, j\}$ and $g^1(C) = \{i^1(C), j^1(C)\}$.

Stage $p+1$. We have defined the sets $A(C)$ and $g^p(C)$. Take an arc $\{i, j\} \in A(C)$ such that $c_{ij} = \min_{\{k,l\} \in A(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Two cases are possible:

1. $g^p(C) \cup \{i, j\}$ has a cycle. Go to the beginning of Stage $p+1$ with $A(C) = A(C) \setminus \{i, j\}$ and $g^p(C)$ the same.

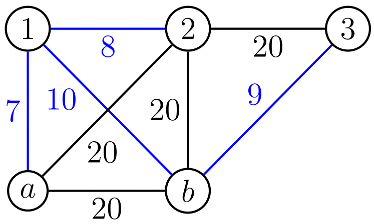
2. $g^p(C) \cup \{i, j\}$ has no cycles. Take $\{i^{p+1}(C), j^{p+1}(C)\} = \{i, j\}$, $A(C) = A(C) \setminus \{i, j\}$ and $g^{p+1}(C) = g^p(C) \cup \{i^{p+1}(C), j^{p+1}(C)\}$. Go to Stage $p+2$.

This process is completed in $|N| + |M| - 1$ stages, exactly the minimum number of arcs that are needed in order to connect all agents with all sources. $g^{|N|+|M|-1}(C)$ is said to be a tree obtained following Kruskal's algorithm (the algorithm leads to a tree which is not always unique). When there is no ambiguity we write A, g^p, and $\{i^p, j^p\}$ instead of $A(C)$, $g^p(C)$, and $\{i^p(C), j^p(C)\}$ respectively.

We denote by $m(N, M, C)$ the cost of any minimal tree in (N, M, C).

Once the minimal tree is obtained, an interesting issue is how to divide its cost among the agents. A *cost allocation rule* is a map ψ that associates a vector of cost shares $\psi(N,M,C) \in \mathbb{R}^N$ with each problem (N,M,C) such that $\sum_{i \in N} \psi_i(N,M,C) = m(N,M,C)$.

Example 1 *Consider the problem (N,M,C) where $N = \{1,2,3\}$, $M = \{a,b\}$, $c_{1a} = 7$, $c_{21} = 8$, $c_{3b} = 9$, $c_{1b} = 10$, and $c_{ij} = 20$ otherwise.*



(The cost of all other arcs is 20.)

The unique minimal tree is $\{\{1,a\},\{1,2\},\{1,b\},\{3,b\}\}$ and $m(N,M,C) = 34$.

3 The folk rule in minimum cost spanning tree problems with multiple sources

In this section we extend various definitions of the folk rule to our setting. The first one is as the Shapley value of the irreducible game, the second is as an obligation rule, the third is as a partition rule, and the fourth is through simple problems.

3.1 The Shapley value of the irreducible game

In *mcstp* Bergantiños and Vidal-Puga (2007a) define the folk rule as the Shapley value of the irreducible game. We now extend this definition to the case of multiple sources.

We define the irreducible problem associated with an *mcstp* following Bird (1976). Let (N,M,C) be a problem and t a minimal tree in (N,M,C). We define the *minimal network* (N,M,C^t) associated with t where $c_{ij}^t = \max_{\{k,l\} \in g_{ij}} \{c_{kl}\}$ and g_{ij} denotes the unique path in t from i to j. It is well-known that C^t is independent of what t is chosen. The *irreducible problem* (N,M,C^*) of (N,M,C) can thus be defined as the minimal network (N,M,C^t) associated with any minimal tree t. C^* is referred to as the *irreducible matrix*.

A *game with transferable utility*, briefly a *TU game*, is a pair (N,v) where $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$.

We associate with each problem (N,M,C) a *TU* game (N, v_{C^*}), called the *irreducible game*. For each $S \subset N$, $v_{C^*}(S) = m(S,M,C^*)$. This means that the value of a coalition is the minimum cost (in C^*) of connecting the agents in S to every source using only the locations of the members in S.

Let Π_N be the set of all permutations over the finite set N. For each $\pi \in \Pi_N$, let $Pre(i, \pi)$ denote the set of agents of N which comes before i in the order π, *i.e.* $Pre(i, \pi) = \{j \in N \text{ such that } \pi(j) < \pi(i)\}$. For each agent $i \in N$, the Shapley value of a TU game (N, v) (Shapley, 1953) is the average of its marginal contributions:

$$Sh_i(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} (v(Pre(i, \pi) \cup \{i\}) - v(Pre(i, \pi)).$$

Definition 1 *For each problem (N, M, C) the rule f^{Sh} is defined as the Shapley value of the irreducible game associated with (N, M, C). Namely, $f^{Sh}(N, M, C) = Sh(N, v_{C^*})$.*

We now compute the Shapley value of the irreducible game in **Example 1**. Since the unique minimal tree is $\{\{1, a\}, \{1, 2\}, \{1, b\}, \{3, b\}\}$, $c^*_{1a} = 7$, $c^*_{12} = 8$, $c^*_{1b} = 10$, and $c^*_{3b} = 9$. Besides, $c^*_{2a} = 8$, and $c^*_{ij} = 10$ otherwise. The irreducible game is as follows:

S	$v_{C^*}(S)$
$\{1\}$	17
$\{2\}$	18
$\{3\}$	19
$\{1, 2\}$	25
$\{1, 3\}$	26
$\{2, 3\}$	27
$\{1, 2, 3\}$	34

Thus,

$$f^{Sh}(N, M, C) = \left(\frac{62}{6}, \frac{68}{6}, \frac{74}{6}\right) = (10.33, 11.33, 12.33).$$

3.2 Obligation rules

Tijs *et al.* (2006) introduce the family of obligation rules for *mcstp.* They are defined through obligation functions. An obligation function is a map o assigning to each $S \in 2^{N_0} \setminus \{\emptyset\}$ a vector $o(S)$ meeting the requirements that (i) if $0 \notin S$, $o(S) \in \Delta(S)$, (ii) if $0 \in S$, $o_i(S) = 0$ for all $i \in S$, and (iii) for each $S, T \in 2^N \setminus \{\emptyset\}$ such that $S \subset T$ and $i \in S$, $o_i(S) \geq o_i(T)$. An obligation function can be seen as follows: Assume that agents in S are connected with one another. They need to construct an arc from any agent in S to the source so that they are all connected. Thus, $o_i(S)$ represents the proportion of the cost of that arc that each agent $i \in S$ must pay. If the agents in S are already connected to the source, i.e. $0 \in S$, then they do not need to construct any arc and so their obligation is zero, $o_i(S) = 0$ for all $i \in S$.

The obligation rule associated with an obligation function o, which is denoted by f^o, is defined through Kruskal's algorithm as follows. The cost of each arc that is constructed at each stage of Kruskal's algorithm is divided among the agents who benefit from its construction. Each agent pays the difference between his obligation to the component to which he belonged before the arc was added and the one afterwards. Tijs *et al.* (2006) prove that the obligation rule f^o is well-defined, namely it is independent of the minimal tree obtained by Kruskal's algorithm.

The folk rule corresponds to the obligation function where for each $S \subset N$ and $i \in S$, $o_i^*(S) = \dfrac{1}{|S|}$.

We now extend this definition to the case of multiple sources. Let $P = \{S_1, ..., S_m\} \in P(N \cup M)$. We assume that for each $S_k \in P$, agents in S_k are connected with one another. We define $o_i(P)$, the obligation of each agent $i \in N$ in P. Note that in *mcstp* if $i \in S_k$, the obligation of agent i depends only on S_k (the element of the partition to which i belongs). Nevertheless in our case this will depend on the whole structure of the partition in connected components.

Consider a link that joins two components of P with sources. Since all agents in N are interested in such a link, all agents have the same obligation over that link.

Consider a link that joins a component S_k with no source ($S_k \cap M = \emptyset$) to a component $S_{k'}$ with sources ($S_{k'} \cap M \neq \emptyset$). Since only agents in S_k are interested in such a link, only agents in S_k have obligations over it. Formally, for each $i \in S_k \cap N$, the obligation function o^* is defined as

$$o_i^*(P) = \begin{cases} \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} & \text{if } S_k \cap M \neq \emptyset, \\ \dfrac{|\{S_j \in P : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \dfrac{1}{|S_k|} & \text{if } S_k \cap M = \emptyset. \end{cases} \tag{1}$$

It is straightforward to see that when there is a single source ($|M| = 1$), o^* coincides with the obligation function associated with the folk rule in *mcstp*.

The obligation rule f^{o^*} associated with the obligation o^* is defined in the same way as in the classical model.

Definition 2 *For each problem* (N, M, C) *and each* $i \in N$ *the rule* f^{o^*} *is defined as*

$$f_i^{o^*}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} [o_i^*(P(g^{p-1})) - o_i^*(P(g^p))].$$

In the next section we prove that f^{o^*} is well-defined, namely for each problem (N, M, C), f^{o^*} divides $m(N, M, C)$ among the agents and is independent of the tree selected by Kruskal's algorithm.

We now compute f^{o^*} in **Example 1**

Arc	$P(g)$	$o_1^*(P(g))$	$o_2^*(P(g))$	$o_3^*(P(g))$
$\emptyset$	$\{1,2,3,a,b\}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$	$\frac{2-1}{3}+\frac{1}{1}=1+\frac{1}{3}$
$\{1,a\}$	$\{1a,2,3,b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$
$\{1,2\}$	$\{12a,3,b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}+1=1+\frac{1}{3}$
$\{3,b\}$	$\{12a,3b\}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$	$\frac{2-1}{3}=\frac{1}{3}$
$\{1,b\}$	$\{123ab\}$	0	0	0

Thus,

$$\begin{aligned}
f_1^{o^*}(N,M,C) &= c_{1a}+\frac{1}{3}c_{1b}=7+\frac{10}{3}=10.33\\
f_2^{o^*}(N,M,C) &= c_{12}+\frac{1}{3}c_{1b}=8+\frac{10}{3}=11.33\\
f_3^{o^*}(N,M,C) &= c_{3b}+\frac{1}{3}c_{1b}=9+\frac{10}{3}=12.33
\end{aligned}$$

3.3 Partition rules

Bergantiños *et al.* (2010, 2011) introduce a family of rules using Kruskal's algorithm. At each step of the algorithm, using the sharing functions, the cost of the selected arc is divided among the agents. A sharing function ϱ is a map that specifies the portion of the cost paid by each agent at each step of Kruskal's algorithm.

We now explain the sharing function inducing the folk rule. Assume that when an arc is added, two components S_k and S_l are joined. The sharing function is defined through the following principles.

1. When a component without the source is joined to one with the source, only agents in the component without the source obtain benefits. Thus, the full cost of the arc is paid by the agents in the component without the source.

2. When two components without the source are joined, agents in both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

Let $i \in S_k$. The proportion of the arc paid by agent i is:

$$\varrho_i(P,P') = \begin{cases} 0 & \text{if } 0 \in S_k, \\ \dfrac{1}{|S_k|} & \text{if } 0 \in S_l, \\ \dfrac{|S_l|}{|S_k \cup S_l||S_k|} & \text{if } 0 \notin S_k \cup S_l. \end{cases}$$

Next we extend this definition of the folk rule to the case of multiple sources. Let $P = \{S_1, ..., S_m\} \in P(N \cup M)$. We assume that for each $S_k \in P$, agents in S_k are connected to one another. Let P' be a partition obtained from P after components S_k and S_l are joined. We define the sharing function ϱ as follows. Cases 1 and 2 are similar to the ones in *mcstp*. Case 3 is new.

1. When we join a component without sources to one with sources, only agents in the component without sources benefit. Thus, the full cost of the arc is paid by the agents in the component without sources.

2. When we join two components without sources, agents of both components benefit. We assume that the total amount paid by one component is proportional to the number of agents in the other. We further assume that all agents in the same component pay the same amount.

3. When we join two components with sources, all agents in the problem benefit. Thus, the cost of that arc is divided equally among all agents in the problem.

Formally, for each $i \in N$ the sharing function ϱ^* is defined as

$$\varrho_i^*(P, P') = \begin{cases} \dfrac{1}{|N|} & \text{if } S_k \cap M \neq \emptyset, S_l \cap M \neq \emptyset, \\ \dfrac{1}{|S_k|} & \text{if } S_k \subseteq N,\ S_l \cap M \neq \emptyset, \text{ and } i \in S_k, \\ \dfrac{|S_l|}{|S_k \cup S_l||S_k|} & \text{if } S_k \cup S_l \subseteq N \text{ and } i \in S_k, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is clear that $\varrho^*(P, P') \in \Delta(N)$.

Definition 3 *For each problem (N, M, C) and each $i \in N$ the partition function sharing rule f^{ϱ^*} is defined as*

$$f_i^{\varrho^*}(N, M, C) = \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p}[\varrho_i^*(P(g^{p-1}), P(g^p))].$$

In the last section we prove that f^{ϱ^*} is a well-defined rule, namely it does not depend on what tree is chosen by Kruskal's algorithm.

We now compute f^{ϱ^*} in **Example 1**

Arc	$P(g^{p-1}), P(g^p)$	$\varrho_1^*(P(g^{p-1}), P(g^p))$	$\varrho_2^*(P(g^{p-1}), P(g^p))$	$\varrho_3^*(P(g^{p-1}), P(g^p))$
$\{1, a\}$	$\{1, a, 2, 3, b\}$ $\{1a, 2, 3, b\}$	1	0	0
$\{1, 2\}$	$\{1a, 2, 3, b\}$ $\{12a, 3, b\}$	0	1	0
$\{3, b\}$	$\{12a, 3, b\}$ $\{12a, 3b\}$	0	0	1
$\{1, b\}$	$\{12a, 3b\}$ $\{123ab\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Thus,

$$\begin{aligned} f_1^{\varrho^*}(N, M, C) &= c_{1a} + \frac{1}{3}c_{1b} = 7 + \frac{10}{3} = 10.33 \\ f_2^{\varrho^*}(N, M, C) &= c_{12} + \frac{1}{3}c_{1b} = 8 + \frac{10}{3} = 11.33 \\ f_3^{\varrho^*}(N, M, C) &= c_{3b} + \frac{1}{3}c_{1b} = 9 + \frac{10}{3} = 12.33. \end{aligned}$$

3.4 The cone-wise decomposition

Norde *et al*. (2004) prove that every *mcstp* can be written as a non-negative linear combination of *simple mcstp* where the costs of the arcs are 0 or 1. Branzei *et al*. (2004) define the folk rule first in simple *mcstp* as follows. Agents connected to the source through a 0 cost path pay nothing. Agents connected with one another through a 0 cost path pay the cost of connecting to the source equally. Later they extend this definition to the general case in a linear way following the result by Norde *et al.* (2004).

We first introduce the folk rule in simple *mcstp* following Branzei *et al*. (2004). Given a simple *mcstp* (N_0, C) and $S \subset N$, we say that i, $j \in N$, $i \neq j$ are (C, S)-*connected* if there exists a path g from i to j satisfying that for all $\{k, l\} \in g$, $c_{kl} = 0$ and $\{k, l\} \subset S$. We say that $S \subset N$ is a C-*component* if two conditions hold. First, for all $i, j \in S$, i and j are (C, S)-*connected*. Second, S is maximal, *i.e.* if $S \subsetneq T$ there exist $i, j \in T$, $i \neq j$ such that i and j are not (C, T)-*connected*. It is obvious that the set of C-components is a partition of N.

Given a simple *mcstp* (N_0, C), the folk rule is defined as follows. Consider $i \in N$ and let S_i be the $C-$ *component* to which i belongs. Thus,

$$f_i(N_0, C) = \begin{cases} \dfrac{1}{|S_i|} & \text{if } c_{0j} = 1 \text{ for all } j \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Namely, agents belonging to a $C-$ *component* which is connected at 0 cost to the source pay nothing, whereas agents belonging to a $C-$ *component* which is connected at a cost of 1 to the source divide this cost equally among the members in the $C-$ *component*.

Lemma 1 adapts the results of Norde *et al.* (2004) to our setting.

Lemma 1 *For each problem (N, M, C), there exist a positive number $m(C) \in \mathbb{N}$, a sequence $\{C^q\}_{q=1}^{m(C)}$ of cost matrices, and a sequence $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:*

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) *For each $q \in \{1, \ldots, m(C)\}$, there exists a network g^q such that $c_{ij}^q = 1$ if $(i,j) \in g^q$ and $c_{ij}^q = 0$ otherwise.*

(3) *Take $q \in \{1, \ldots, m(C)\}$ and $\{i,j,k,l\} \subset N_0$. If $c_{ij} \leq c_{kl}$, then $c_{ij}^q \leq c_{kl}^q$.*

Branzei *et al.* (2004) extend the definition of the folk rule to a general *mcstp* (N_0, C) using Lemma 1. Namely, the folk rule is defined as

$$\sum_{q=1}^{m(C)} x^q f\left(N_0, C^q\right)$$

where $f\left(N_0, C^q\right)$ denotes the folk rule in the simple *mcstp* (N_0, C^q).

Since we have multiple sources we need to adapt the procedure. The first change is related to the definition of $C-$ *components*. Instead of considering each component as a subset of the set of agents N, we now consider a $C-$ *component* as a subset of $N \cup M$.

Let (N, M, C) be a simple problem. Denote by $P = \{S_1, \ldots, S_r\}$ the set of $C-$ *components*. The rule f^{CW} for simple problems is defined as follows: We first connect each component with no sources to a component with sources and divide the cost equally among the agents in the component. Then we connect the components with sources with one another and divide the cost equally among all the agents. Formally, consider $i \in N$ and let S_i be the $C-$ *component* to which i belongs. Thus,

$$f_i^{CW}(N, M, C) = \begin{cases} \dfrac{|S_j \in P : S_j \cap M \neq \emptyset| - 1}{|N|} & \text{if } S_i \cap M \neq \emptyset, \\ \dfrac{1}{|S_i|} + \dfrac{|S_i \in P : S_i \cap M \neq \emptyset| - 1}{|N|} & \text{if } S_i \cap M = \emptyset. \end{cases}$$

Definition 4 *For each problem (N, M, C) and each $i \in N$ the cone-wise rule f^{CW} is defined as*

$$f_i^{CW}(N, M, C) = \sum_{q=1}^{m(C)} x^q f_i^{CW}(N, M, C^q).$$

We now compute f^{CW} in **Example 1**.

Note that $C = \sum_{q=1}^{5} x^q C^q$ where $x^1 = 7$, $x^2 = x^3 = x^4 = 1$, $x^5 = 10$ and

Arcs	C^1	C^2	C^3	C^4	C^5
$\{a,1\}$	1	0	0	0	0
$\{1,2\}$	1	1	0	0	0
$\{b,3\}$	1	1	1	0	0
$\{b,1\}$	1	1	1	1	0
$\{a,b\}$	1	1	1	1	1
$\{a,2\}$	1	1	1	1	1
$\{a,3\}$	1	1	1	1	1
$\{b,2\}$	1	1	1	1	1
$\{1,3\}$	1	1	1	1	1
$\{2,3\}$	1	1	1	1	1

We compute $f^{CW}(N, M, C^q)$ for each $q = 1, ..., 5$.

1. $C^1 - \textit{components}$ are $\{1, 2, 3, a, b\}$.
$$f^{CW}(N, M, C^1) = \left(1 + \frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

2. $C^2 - \textit{components}$ are $\{a1, 2, 3, b\}$.
$$f^{CW}(N, M, C^2) = \left(\frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right).$$

3. $C^3 - \textit{components}$ are $\{a12, 3, b\}$.
$$f^{CW}(N, M, C^3) = \left(\frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3}\right).$$

4. $C^4 - \textit{components}$ are $\{a12, b3\}$.
$$f^{CW}(N, M, C^4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

5. $C^5 - \textit{components}$ are $\{ab123\}$.
$$f^{CW}(N, M, C^5) = (0, 0, 0).$$

Then,

$$\begin{aligned}
f^{CW}(N, M, C) &= \sum_{q=1}^{5} x^q f^{CW}(N, M, C^q) \\
&= 7\left(1 + \frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right) + \left(\frac{1}{3}, 1 + \frac{1}{3}, 1 + \frac{1}{3}\right) \\
&\quad + \left(\frac{1}{3}, \frac{1}{3}, 1 + \frac{1}{3}\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + 10\,(0, 0, 0) \\
&= (10.33, 11.33, 12.33)\,.
\end{aligned}$$

4 Results

We now present our main results. In Proposition 1 we prove that the obligation rule f^{o^*} and the Kruskal sharing rule f^{ϱ^*} are well-defined. In Theorem 1 we prove that all four definitions of the rule result in the same allocation.

Proposition 1 *f^{o^*} and f^{ϱ^*} are well-defined.*

Proof. We need to prove two statements. First, f^{o^*} and f^{ϱ^*} divide the cost of the minimal tree $m(N,M,C)$ among the agents. Second, the definition of f^{o^*} and f^{ϱ^*} does not depend on what minimal tree is chosen by Kruskal's algorithm.

We start with f^{o^*}. In order to prove that f^{o^*} divides $m(N,M,C)$ among the agents, it suffices to prove that for each $p=1,\ldots,|N|+|M|-1$, the cost of arc $\{i^p,j^p\}$ is allocated in full among the agents in N.

Given $P=\{S_1,...,S_m\}\in P(N\cup M)$, it is trivial to see that $\sum\limits_{i\in N} o_i^*(P)=m-1$. Then,

$$\begin{aligned}\sum_{i\in N}[o_i^*(P(g^{p-1}))-o_i^*(P(g^p))] &= \sum_{i\in N} o_i^*(P(g^{p-1}))-\sum_{i\in N} o_i^*(P(g^p))\\ &= |P(g^{p-1})|-1-(|P(g^p)|-1)\\ &= |P(g^{p-1})|-|P(g^p)|=1.\end{aligned}$$

Next we prove that f^{o^*} does not depend on which minimal tree is chosen by Kruskal's algorithm.

Given a tree $t=\{\{i^p,j^p\}\}_{p=1}^{|N|+|M|-1}$ obtained by Kruskal's algorithm, we define the following:

- $B^0(t)=\emptyset$, $c^0(t)=c^0=0$.
- $c^1(t)=\min\limits_{\{k,l\}\in t\setminus B^0(t)}\{c_{kl}\}$, $c^1=\min\limits_{\{k,l\}\subset N\cup M, c_{kl}>c^0}\{c_{kl}\}$, and $B^1(t)=\{\{i,j\}\in t: c_{ij}=c^1(t)\}$.
- In general, $c^p(t)=\min\limits_{\{k,l\}\in t\setminus\cup_{q=0}^{p-1}B^q(t)}\{c_{kl}\}$, $c^p=\min\limits_{\{k,l\}\subset N\cup M, c_{kl}>c^{p-1}}\{c_{kl}\}$ and $B^p(t)=\{\{i,j\}\in t: c_{ij}=c^p(t)\}$.

This process ends when we find $m(t)\le|N|+|M|-1$ such that $\cup_{p=0}^{m(t)-1}B^p(t)\subsetneq t=\cup_{p=0}^{m(t)}B^p(t)$. Note that $m(t)$ denotes the number of arcs in t with different cost.

By Kruskal's algorithm, for all $r=1,...,m(t)$, $c^r(t)=c^r$.

Next we prove that $P(B^1(t))=P(\{(i,j):c_{ij}\le c^1\})$. Since $B^1(t)\subset\{\{i,j\}:c_{ij}\le c^1\}$, $P(B^1(t))$ is finer than $P(\{(i,j):c_{ij}\le c^1\})$. Suppose that $P(B^1(t))\neq P(\{(i,j):c_{ij}\le c^1\})$. Then, there exist $S,S'\in P(B^1(t))$, $S\neq S'$, $k\in S$, and $l\in S'$ such that $c_{kl}\le c^1$. Thus, $B^1(t)\cup\{\{k,l\}\}$ has no cycles and $\{k,l\}\notin t$, which contradicts the construction of t via Kruskal's algorithm. Then, $P(B^1(t))=P(\{\{i,j\}:c_{ij}\le c^1\})$.

Suppose now that for all $r < q$,

$$P\left(\cup_{p=0}^{r} B^p(t)\right) = P(\{\{k,l\} : c_{kl} \leq c^r\}).$$

Using arguments similar to those used in the case $q = 1$, we can prove that

$$P(\cup_{p=0}^{q} B^p(t)) = P(\{(i,j) : c_{ij} \leq c^q\}).$$

Since $t = \cup_{q=1}^{m(t)} B^q(t)$ and $c_{ij} = c^q$ for all $\{i,j\} \in B^q(t)$ and all $q = 0, \ldots, m(t)$,

$$\begin{aligned}
f_i^{o^*}(N,M,C) &= \sum_{p=1}^{|N|+|M|-1} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))\right] \\
&= \sum_{q=1}^{m(t)} \left(\sum_{p=|\cup_{s=0}^{q-1} B^s(t)|+1}^{|\cup_{s=0}^{q} B^s(t)|} c_{i^p j^p} \left[o_i^*(P(g^{p-1})) - o_i^*(P(g^p))\right] \right) \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(g^{|\cup_{s=0}^{q-1} B^s(t)|})) - o_i^*(P(g^{|\cup_{s=0}^{q} B^s(t)|}))\right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\cup_{p=0}^{q-1} B^p(t))) - o_i^*(P(\cup_{p=0}^{q} B^p(t)))\right] \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\} : c_{ij} \leq c^{q-1}\})) - o_i^*(P(\{\{i,j\} : c_{ij} \leq c^q\}))\right]. \quad (3)
\end{aligned}$$

Thus, f^{o^*} does not depend on the minimal tree t.

To prove that f^{ϱ^*} is well-defined, it is enough to show that at each stage p of Kruskal's algorithm and for each agent $i \in N$,

$$\varrho_i^*(P(g^{p-1}), P(g^p)) = o_i^*(P(g^{p-1})) - o_i^*(P(g^p)).$$

Assume without loss of generality that $g^p = g^{p-1} \cup \{k,l\}$, $P(g^{p-1}) = \{S_1, \ldots, S_r\}$, $k \in S_1$, $l \in S_2$ and $P(g^p) = \{S_2', \ldots, S_r'\}$ where $S_2' = S_1 \cup S_2$ and $S_j' = S_j$ for each $j = 3, \ldots, r$. We consider several cases:

1. $S_1 \cup S_2 \subset N$:

   (a) $i \notin S_2'$. Since $S_i' = S_i$ it is trivial to see that

   $$o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) = 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)).$$

   (b) $i \in S_2'$. Assume $i \in S_1$ (since the other case is analogue, we omit it). Then,

   $$\begin{aligned}
   o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_1|} - \frac{1}{|S_1 \cup S_2|} = \frac{|S_2|}{|S_1 \cup S_2||S_1|} \\
   &= \varrho_i^*(P\left(g^{p-1}\right), P(g^p)).
   \end{aligned}$$

2. $S_1 \cap M \neq \emptyset$ and $S_2 \cap M \neq \emptyset$:

   (a) $i \notin S_2'$ and $S_i \subset N$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_i'|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} - \frac{1}{|S_i|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

   (b) $i \notin S_2'$ and $S_i \cap M \neq \emptyset$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

   (c) $i \in S_2'$. Assume $i \in S_1$ (since the other case is analogous, we omit it). Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|N|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

3. $S_1 \subset N$ and $S_2 \cap M \neq \emptyset$ (since the case $S_1 \cap M \neq \emptyset$ and $S_2 \subset N$ is similar, we omit it):

   (a) $i \notin S_2'$ and $S_i \subset N$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{1}{|S_i'|} - \frac{1}{|S_i|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

   (b) $i \notin S_2'$ and $S_i \cap M \neq \emptyset$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(c) $i \in S_2' \cap S_1$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} + \frac{1}{|S_1|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= \frac{1}{|S_1|} = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

(d) $i \in S_2' \cap S_2$. Then,

$$\begin{aligned} o_i^*(P(g^{p-1})) - o_i^*(P(g^p)) &= \frac{|\{S_j \in P(g^{p-1}) : S_j \cap M \neq \emptyset\}| - 1}{|N|} \\ &\quad - \frac{|\{S_j' \in P(g^p) : S_j' \cap M \neq \emptyset\}| - 1}{|N|} \\ &= 0 = \varrho_i^*(P\left(g^{p-1}\right), P(g^p)). \end{aligned}$$

■

Theorem 1 *The four rules that we have defined are the same, namely for each problem (N, M, C), we have that*

$$f^{Sh}(N, M, C) = f^{o^*}(N, M, C) = f^{\varrho^*}(N, M, C) = f^{CW}(N, M, C).$$

Proof. From the proof of Proposition 1 we know that $f^{o^*} = f^{\varrho^*}$. We now prove that $f^{Sh} = f^{CW}$ and $f^{\varrho^*} = f^{CW}$.

We first prove that f^{CW} and f^{Sh} coincide in simple problems. Let (N, M, C) be a simple problem. Let $\{S_1, \ldots, S_k\}$ be the set of *C-components*. For each $i \in N \cup M$ let S_i be the *C-component* to which i belongs. Let t be a minimal tree. It is easy to prove that all the elements inside a component are connected at zero cost in t, while the components connect to one another through arcs of cost 1. Note that in the irreducible problem (N, M, C^*) we have that $c_{ij}^* = 0$ when $S_i = S_j$ while $c_{ij}^* = 1$ when $S_i \neq S_j$. Thus, the set of *C-components* and *C^*-components* coincide.

Recall that for each $i \in N$,

$$\begin{aligned} f_i^{CW}(N, M, C) &= \begin{cases} \dfrac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|} & \text{if } S_i \cap M \neq \emptyset, \\ \dfrac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|} + \dfrac{1}{|S_i|} & \text{otherwise.} \end{cases} \\ f_i^{Sh}(N, M, C) &= Sh_i(N, v_{C^*}) = \frac{1}{|N|!} \sum_{\pi \in \Pi} (v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi))). \end{aligned}$$

We consider two cases:

1. $S_i \cap M \neq \emptyset$. Given an order $\pi \in \Pi$, if $\pi(i) = 1$, agent i has to pay the cost of connecting its component to all sources. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = |S_k : S_k \cap M \neq \emptyset| - 1$. If $\pi(i) > 1$, this means that when this agent arrives all the components with sources are already connected. Thus, $v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)) = 0$. Therefore,

$$\begin{aligned} f_i^{Sh}(N,M,C) &= \frac{1}{|N|!}\sum_{\pi\in\Pi}(v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) \\ &= \frac{1}{|N|!}\sum_{\pi\in\Pi:\pi(i)=1} |S_k : S_k \cap M \neq \emptyset| - 1 \\ &= \frac{1}{|N|!}(|N|-1)!\,(|S_k : S_k \cap M \neq \emptyset| - 1) \\ &= \frac{|S_k : S_k \cap M \neq \emptyset| - 1}{|N|} = f_i^{CW}(N,M,C). \end{aligned}$$

2. $S_i \cap M = \emptyset$. Given an order $\pi \in \Pi$, we compute $v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi))$ distinguishing several cases.

   (a) $Pre(i,\pi) \cap S_i \neq \emptyset$. Thus,
   $$v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) = 0.$$

   (b) $Pre(i,\pi) \cap S_i = \emptyset = Pre(i,\pi)$. Then $\pi(i) = 1$. Thus,
   $$v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) = |S_k : S_k \cap M \neq \emptyset|.$$

   (c) $Pre(i,\pi) \cap S_i = \emptyset \neq Pre(i,\pi)$. In this case $\pi(i) > 1$. Thus,
   $$v_{C^*}(Pre(i,\pi)\cup\{i\}) - v_{C^*}(Pre(i,\pi)) = 1.$$

   Let
   $$\Pi^* = \{\pi \in \Pi : Pre(i,\pi) \cap S_i = \emptyset \text{ and } \pi(i) > 1\}.$$
   Taking into account the computations above we have that
   $$f_i^{Sh}(N,M,C) = \frac{1}{|N|}|S_k : S_k \cap M \neq \emptyset| + \frac{1}{|N|!}|\Pi^*|.$$

   We have that
   $$\frac{1}{|N|!}|\Pi^*| = \frac{1}{|N|!}\sum_{k=1}^{|N|-|S_i|}\frac{(|N|-|S_i|)!}{(|N|-|S_i|-k)!}(|N|-k-1)!.$$

   We consider $|S_i| = m+1$. Then,
   $$\begin{aligned} \frac{1}{|N|!}|\Pi^*| &= \sum_{k=1}^{|N|-m-1}\frac{(|N|-m-1)!(|N|-k-1)!}{(|N|-m-k-1)!|N|!} \\ &= \frac{(|N|-m-1)!m!}{|N|!}\sum_{k=1}^{|N|-m-1}\binom{|N|-k-1}{m}. \end{aligned}$$

Since

$$\begin{aligned}\binom{x+1}{y+1}-\binom{x}{y+1} &= \frac{(x+1)!}{(y+1)!\,(x-y)!}-\frac{x!}{(y+1)!\,(x-y-1)!}\\ &= \frac{[(x+1)-(x-y)]\,x!}{(y+1)!\,(x-y)!}\\ &= \frac{x!}{y!\,(x-y)!}=\binom{x}{y}\end{aligned}$$

we have that

$$\begin{aligned}\sum_{k=1}^{|N|-m-1}\binom{|N|-k-1}{m} &= \sum_{k=1}^{|N|-m-2}\binom{|N|-k-1}{m}+\binom{m}{m}\\ &= \sum_{k=1}^{|N|-m-2}\left[\binom{|N|-k}{m+1}-\binom{|N|-k-1}{m+1}\right]+\binom{m}{m}\\ &= \binom{|N|-1}{m+1}-\binom{m+1}{m+1}+\binom{m}{m}\\ &= \binom{|N|-1}{m+1}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{|N|!}|\Pi^*| &= \frac{(|N|-m-1)!m!}{|N|!}\binom{|N|-1}{m+1}\\ &= \frac{(|N|-m-1)!m!}{|N|!}\frac{(|N|-1)!}{(m+1)!\,((|N|-m-2)!)}\\ &= \frac{|N|-m-1}{|N|(m+1)}=\frac{1}{m+1}-\frac{1}{|N|}=\frac{1}{|S_i|}-\frac{1}{|N|}.\end{aligned}$$

Thus,

$$\begin{aligned}f_i^{Sh}(N,M,C) &= \frac{|S_k : S_k\cap M\neq\emptyset|}{|N|}+\frac{1}{|S_i|}-\frac{1}{|N|}\\ &= \frac{|S_k : S_k\cap M\neq\emptyset|-1}{|N|}+\frac{1}{|S_i|}=f_i^{CW}(N,M,C).\end{aligned}$$

Consider now a general problem (N,M,C) and $i\in N$. Thus,

$$f_i^{CW}(N,M,C)=\sum_{q=1}^{m(C)}x^q f_i^{CW}(N,M,C^q)=\sum_{q=1}^{m(C)}x^q Sh_i(N,v_{(C^q)^*}).$$

Since the Shapley value satisfies additivity on v,

$$\sum_{q=1}^{m(C)}x^q Sh_i(N,v_{(C^q)^*})=Sh_i\left(N,v_{\sum_{q=1}^{m(C)}x^q(C^q)^*}\right).$$

Now it remains to prove that $C^* = \sum_{q=1}^{m(C)} x^q (C^q)^*$. Let t be a minimal tree and g_{ij} be the unique path in t from i to j. We know that $c_{ij}^* = \max_{\{k,l\} \in g_{ij}} \{c_{kl}\} = c_{i'j'}$. By Lemma 1, the order of the arcs according to its cost is preserved in each C^q. So t is also a minimal tree for each simple problem C^q. Thus, $c_{ij}^{q*} = \max_{\{k,l\} \in g_{ij}} \{c_{kl}^q\} = c_{i'j'}^q$ and hence

$$c_{ij}^* = c_{i'j'} = \sum_{q=1}^{m(C)} x^q c_{i'j'}^q = \sum_{q=1}^{m(C)} x^q c_{ij}^{q*}.$$

We now prove that f^{o^*} coincides with f^{CW}. Let (N, M, C) be a problem.

We define t, $m(t)$ and c^k $(k = 1, ..., m(t))$ as in the proof of Proposition 1 when we have proved that f^{o^*} does not depend on which minimal tree is chosen by Kruskal's algorithm.

By Lemma 1, $C = \sum_{q=1}^{m(C)} x^q C^q$. Besides, by Norde *et al.* (2004), $c^1 = \min\{c_{ij} : c_{ij} > 0\}$ and

$$c_{ij}^1 = \begin{cases} 0 & \text{when } c_{ij} < c^1, \\ 1 & \text{when } c_{ij} \geq c^1. \end{cases}$$

In general,

$$c^q = \min\{c_{ij} : c_{ij} > c^{q-1}\},$$

$$c_{ij}^q = \begin{cases} 0 & \text{when } c_{ij} < c^q, \\ 1 & \text{when } c_{ij} \geq c^q, \end{cases}$$

and

$$x^q = \begin{cases} c^1 & \text{when } q = 1, \\ c^q - c^{q-1} & \text{when } q > 1. \end{cases}$$

For each $q = 1, ..., m(C)$ the set of C^q-components coincide with $P(\{\{i,j\} : c_{ij} \leq c^{q-1}\})$.

Obviously $m(t) \leq m(C)$ and t is a minimal tree in C^q for each $q = 1, ..., m(C)$. Besides, for each $q > m(t)$ and each $\{i,j\} \in t$, $c_{ij}^q = 0$. Thus, by definition of f^{o^*}, for each $i \in N$ and each $q = m(t)+1, ..., m(C)$, $f_i^{CW}(N, M, C^q) = 0$. Then,

$$f^{CW}(N, M, C) = \sum_{q=1}^{m(C)} x^q f^{CW}(N, M, C^q) = \sum_{q=1}^{m(t)} x^q f^{CW}(N, M, C^q).$$

By definition of o^* and f^{CW}, for each $i \in N$ and each $q = 1, ..., m(t)$,

$$f_i^{CW}(N, M, C^q) = o_i^*(P(\{\{i,j\} : c_{ij} \leq c^{q-1}\}))$$

where we denote $c^0 = 0$.

Thus,

$$
\begin{aligned}
f_i^{CW}(N,M,C) &= \sum_{q=1}^{m(t)} x^q f_i^{CW}(N,M,C^q) = \sum_{q=1}^{m(t)} x^q o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q-1}\})) \\
&= c^1 o_i^*(P(\{\{i,j\}:c_{ij}\leq c^0\})) + \sum_{q=2}^{m(t)} (c^q - c^{q-1}) o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q-1}\})) \\
&= \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q-1}\})) - o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q}\})) \right] \\
&\quad + c^{m(t)} o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{m(t)}\})).
\end{aligned}
$$

Since $P(\{\{i,j\}:c_{ij}\leq c^{m(t)}\}) = \{N\cup M\}$, for each $i\in N$,

$$o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{m(t)}\})) = 0.$$

So,

$$f_i^{CW}(N,M,C) = \sum_{q=1}^{m(t)} c^q \left[o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q-1}\})) - o_i^*(P(\{\{i,j\}:c_{ij}\leq c^{q}\})) \right].$$

By (3), we deduce that $f_i^{CW}(N,M,C) = f_i^{o^*}(N,M,C)$. ■

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